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Connecting the Tangent Function to Cardinality: A Method for Introducing Set Theory to High School Students

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ABSTRACT High school students often ask questions about the nature of infinity. When contemplating what the "largest number" is, or discussing the speed of light, students bring their own ideas about infinity and asymptotes into the conversation. These are popular ideas, but formal ideas about the nature of mathematical sets, or "set theory," are generally unknown to high school students. The authors propose a method for introducing basic ideas in set theory to high school trigonometry students by connecting prior knowledge of the tangent function and the unit circle to Georg Cantor's ideas about infinity. By doing so, high school teachers have an opportunity to inspire their students with rich mathematics.

KEYWORDS *algebra II, asymptote, cardinality, connecting representations, Georg Cantor, infinity, tangent function*

Introduction

Teachers of Common Core Algebra II courses often struggle to cover the scope of the curriculum due to time constraints. The race to finish the curriculum does little to spark students' interest or engagement in mathematics, although modern teaching philosophies suggest "…a movement towards the student being invited to act like a mathematician instead of passively taking in math" (Hartnett, 2017, para. 6). Rather, "…the interested student should be exposed to mathematics outside the core curriculum, because the standard curriculum is not designed for the top students" (Rusczyk, 2016, para. 2).

Inquisitive mathematics students are often heard discussing infinity, asking questions like, "What is infinity plus infinity?," "Is infinity a number or an idea?," "What is infinity plus one?," and "Can one infinity be bigger than another?" Research has shown that students as young as five or six have a vague notion of infinity as an unlimited process, and mathematics educators have attempted to convey the distinction between two different types of conceptualizations. There is a *potential infinity*, such as continually counting from 1, 2, 3, … etc., which is usually the first encounter of infinity for children, and there is a more nuanced concept of an *actual infinity*, which describes a more concrete mathematical entity. This more advanced viewpoint extends the earlier concept of infinitely counting because it "requires us to conceptualize the potentially infinite process of counting more and more numbers as if it was somehow finished" (Pehkonen, Hannula, Maijala, & Soro, 2006, p. 345). While the Common Core State Standards for Mathematics include extension standards that invite investigation and discovery (e.g., "CCSS.MATH.CONTENT.HSF.TF.B.6: (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed" [National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010]), the lack of time in the classroom more often than not restricts engagement in topics that will not be covered on an end-of-year exam.

Cardinal arithmetic can be used to show that the number of points on the real line is equal to the number of points on any segment of that line. The authors of this manuscript, two experienced high school mathematics teachers, conjectured that this notion is highly counterintuitive and baffling for high school students. The authors interviewed 28 students in a group setting in a Common Core Algebra II class in a Title I, New York City public high school. The untracked class consisted of students with varying achievement levels in mathematics. The students in the class unequivocally demonstrated interest and enthusiasm in discussing the nature of infinity. Students' responses to the question "Can you describe infinity?" are summarized below:

- Infinity is an idea that there is an unlimited amount of numbers going from negative to positive.
- Infinity can be beyond time.
- It's a number that never ends; it's an endless number that never stops going, and never stops growing.
- It's not really a number but more of an idea, because a number is just one singular thing and infinity isn't.
- Infinity is not a number, it's like an idea, because the rules for regular numbers don't apply to it. For example, I once heard someone say that infinity times infinity wouldn't be infinity squared, it would stay infinity.

The students even generated their own questions, such as:

- Is it true what he said, that infinity times infinity is infinity?
- Is infinity minus infinity equal to 0?

The authors saw clear evidence that students were interested in the notion of infinity, and the conversation inspired responses even from students who normally remained quiet during class. While it is exciting for teachers to engage in these discussions with their students, topics in set theory are seemingly so far outside of the curriculum that time does not allow for such activities.

Inspired by their interactions with this Common Core Algebra II class, the authors propose enrichment activities in this manuscript. The authors co-developed these ideas based on their teaching experiences and their desires to inspire students with rich mathematics. This manuscript is not a research study, but rather a report intended to propose ways of exposing high school students to some advanced ideas about set theory and infinity and help them reach surprising conclusions along the way.

Prior Knowledge

Before engaging students in the enrichment activities that will be outlined in this manuscript, it is assumed that students will have learned the following topics:

- Similar triangles;
- Functions (including one-to-one functions) and their features (including asymptotes, end behavior, and intervals of increase/decrease);
- Domain and range;
- Interval notation;
- Definitions of the sine, cosine, and tangent of an angle;
- Unit circle, and the fact that for a point (x, y) on the unit circle, $(x, y) = (\cos\theta, \sin\theta)$;
- Quotient identity: $\frac{(\sin\theta)}{(\cos\theta)} = \tan\theta$;
- Unwrapping the unit circle to graph periodic functions (namely, $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$); and
- Visualization of $\sin\theta$ and $\cos\theta$ on an inscribed right triangle in the unit circle as the lengths of the vertical and horizontal legs, respectively.

Specifically, right before facilitating these enrichment activities, the students should learn how to graph the tangent function on the interval $\left[0, \frac{\pi}{2}\right)$ (see Figure 1).

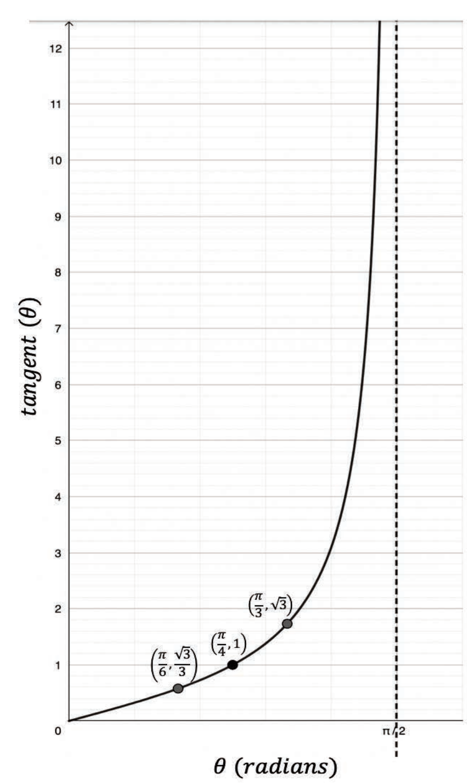

Figure 1. The tangent function on the interval $\left[0, \frac{\pi}{2}\right)$.

Development

To begin our enrichment activities, we will pose the following question to students: Which interval contains more numbers: $\left[0, \frac{\pi}{2}\right)$ or $[0, \infty)$? The students will all think the answer is $[0, \infty)$. However, the following sequence of questions can show students that these sets are in fact equal in size, which they will likely find baffling. Desired responses are indicated in italics.

- Is the tangent function strictly increasing on $\left[0, \frac{\pi}{2}\right)$? *Yes.*
- What is the range of the tangent function on the domain $\left[0, \frac{\pi}{2}\right)$? *$[0, \infty)$.*
- What does the combination of these facts tell us? *The tangent of every angle on $\left[0, \frac{\pi}{2}\right)$ is some real number on $[0, \infty)$. Conversely, every real number on $[0, \infty)$ is the tangent of an angle on $\left[0, \frac{\pi}{2}\right)$.*
- What can we conclude? *Since the tangent function is strictly increasing, every input has exactly one output and every output corresponds to exactly one input. Therefore, the intervals $\left[0, \frac{\pi}{2}\right)$ and $[0, \infty)$ are "equal in size"—they do not have the same length, but they do have equal cardinalities.*

Students may be so overwhelmed by the counterintuitive nature of this idea that they may not "buy it" at first. We propose the following activities to help students understand cardinality and convince themselves that this is, in fact, true.

First, gather the students in the hallway and take them to a classroom they are unfamiliar with. Ask them to devise a way, *without counting,* to determine whether there are the same number of chairs in the room as there are students. Students will surely start sitting down in the chairs—and if every student has one chair, and every chair has one student, they will arrive at the conclusion that there is the same number of chairs as there are students. This leads naturally to the idea that two sets have the same size (or cardinality) if they are in 1–1 correspondence with each other, that is, *if their elements can be matched up.*

Teachers must keep in mind that it is easy to mix up the words "size" and "length" when comparing and contrasting intervals and sets of numbers. It is therefore important to keep reminding students that when we talk about two sets having the same size, we mean that their elements can be paired up until each set is exhausted of its elements. Students will probably have no trouble accepting this, since it is the same way we can tell if two finite sets have the same size (which is exactly what they did when they filled the chairs in the classroom).

Having accepted this, we can next ask if the set of natural numbers is the same size as the set of even numbers. Initially, students will probably all assert that the set of natural numbers is "twice the size" of the set of even numbers. Then, they can be reminded of the definition they agreed upon for "same size" (matching of elements) and can be asked to think about a potential matching between the two sets. Students should come up with the following idea:

natural numbers		even numbers
1	→	2
2	→	4
3	→	6
4	→	8
5	→	10

In other words, every natural number matches with its double. For students who don't initially grasp the idea that {1,2,3,4...} has the same size as {2,4,6,8,...}, teachers can ask: *If the elements in a set are multiplied by two, does the new set hold a different number of elements, or just bigger elements?* This concrete example can be used to show students for the first time that two infinite sets that appear visually different can have the same size (according to the definition that they agreed upon). This exercise can give them a sense of how powerful the notion of an infinite set can be, and a sense of the mystery of infinite sets.

Immediately following, students can be asked, "Are the set of whole numbers and the set of positive integers the same size?" While initially thinking that whole numbers have "one more element" than the set of positive integers (the element 0), students may be inclined to think of a potential matching between these two sets after having seen the previous examples. Hopefully, students will arrive at the relationship which matches each of the whole numbers with its successor in the set of positive integers. After a few of these examples, students should be willing to believe that different infinite sets can have the same size.

We are about to transition from sizes of discrete sets like the integers, to sizes of intervals. Before doing so, it is important to set up a more mathematically formal definition of matching with the students: Two sets have the same size if there is a bijection between them. The bijection matches x from set A, with $f(x)$ from set B, creating a 1–1 mapping. For instance, the prior two examples can be described by the functions $f(n) = 2 \cdot n$ and $f(n) = n + 1$, respectively. So, the level of the discussion is being raised to include functions. Once students accept the notion that a bijective function between two sets can be used to illustrate a mapping between them, we can ask:

Is a strictly increasing function (or a strictly decreasing function) a bijection between its domain and range? How do you know?

The answer is yes, because if a function is strictly increasing or strictly decreasing it must be one-to-one, meaning that in addition to each input having only one output, each output only corresponds to one input (again relating back to the students and chairs example and the concept of a bijection). We can therefore conclude that for a strictly increasing or strictly decreasing function, the domain and range are the same size. This important result will help students who were hesitant to accept the tangent function example from earlier.

We can bring this example back into focus by again asking:

Does the interval $\left[0, \frac{\pi}{2}\right)$ have the same size as $[0, \infty)$? For students who initially argued "no", we now have a concrete definition that will challenge their thinking and demonstrate that the two seemingly very different sets do, in fact, have the same size.

We can ask the students to think of a graph whose domain on $\left[0, \frac{\pi}{2}\right)$ has a range of $[0, \infty)$. Well, the tangent function is one such graph (as demonstrated in Figure 1). Furthermore, we can use our definition of matching because *the tangent function is strictly increasing on this interval.* Therefore, by the students' accepted definition of "matching," since this function is strictly increasing, its domain and range must be equal in size. Therefore, $\left[0, \frac{\pi}{2}\right)$ is the same size as $[0, \infty)$. Wow!

To take it a step further, you can ask students to compare the sizes of the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the real line, $(-\infty, \infty)$. The students will probably be excited at this point by the previous examples and refrain from incorrectly blurting out that these intervals must have different sizes. They will likely be inspired to come up with their own function to illustrate why these intervals do have the same size—and the astute student might notice that the tangent function does it again! Simply extending the graph in Figure 1 to include negative angles, the tangent function is strictly increasing on the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and therefore matches 1–1 with the range $(-\infty, \infty)$.

This will open students' eyes to the idea that two sets or intervals of finite and infinite extent can actually have the same size. Later in their mathematical studies, students will eventually see examples such as $[0, 1]$ matching with $[0, 1)$ and will then be able to generalize these results and conclude that *any interval has the same size as any other interval, regardless of their lengths and regardless of whether one is open, closed, half open, or infinite.*

In addition to the aforementioned method of convincing students why $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ has the same size as $(-\infty, \infty)$, there is a geometric and visual way to demonstrate why this is true. This can be executed as a discovery learning task in a high school classroom and is detailed below.

Most pre-calculus books show the reason that the tangent function is called the "tangent" function can be understood by doing the following:

Given a coordinate plane with a unit circle and an inscribed right triangle in Quadrant I on it (Figure 2), perform the following sequence of steps. Desired student responses are indicated via italicized font and in the diagrams following the prompts (Figure 3).

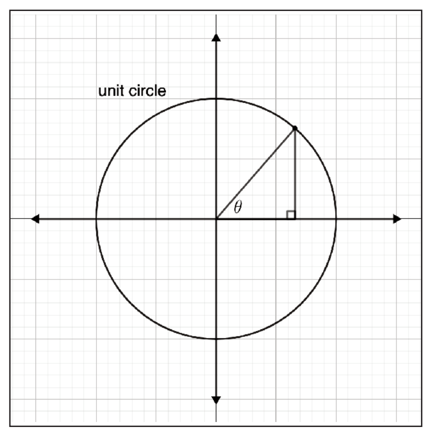

Figure 2. Coordinate plane with a unit circle and inscribed right triangle in Quadrant I.

1. Examine the right triangle that was drawn for you. Label its legs in terms of trigonometric functions of θ, which is the acute angle formed by the radius of the unit circle and the *x-axis.*
2. On the diagram, add the graph of the line $x = 1$. What type of line is this, in relation to the unit circle? *$x = 1$ is tangent to the unit circle. The line intersects the circle at only the point (1, 0).*
3. What kind of angle is formed by the line $x = 1$ and the *x-axis?* Label it on the diagram. *This angle must be a right angle because the segment connecting* (0, 0) *and* (1, 0) *(which is a radius of the unit circle) is horizontal, and* $x = 1$ *is vertical. A tangent to a circle is perpendicular to the radius drawn at the point of tangency.*

4. Extend the hypotenuse of the right triangle so that it meets the line $x = 1$.
5. Locate the vertical segment connecting (1, 0) and the point of intersection of the extended hypotenuse and $x = 1$. Since this vertical segment has an unknown length, label it *"a."*
6. Separately draw the two right triangles that are now on your picture, copying all information that is known about their side lengths and angles.
7. What do you notice about these triangles? Write a proportion relating the legs of the triangles. Simplify the proportion. *The triangles are similar, since they have two equal angles. The proportion relating their legs is* $\frac{\sin\theta}{\cos\theta} = \frac{a}{1}$*. Simplifying yields* $\tan\theta = a$.
8. What can we conclude? *The tangent of an angle in standard position is equal to the length of the vertical segment connecting the point* (1, 0) *and the point of intersection of the terminal ray with the line* $x = 1$.

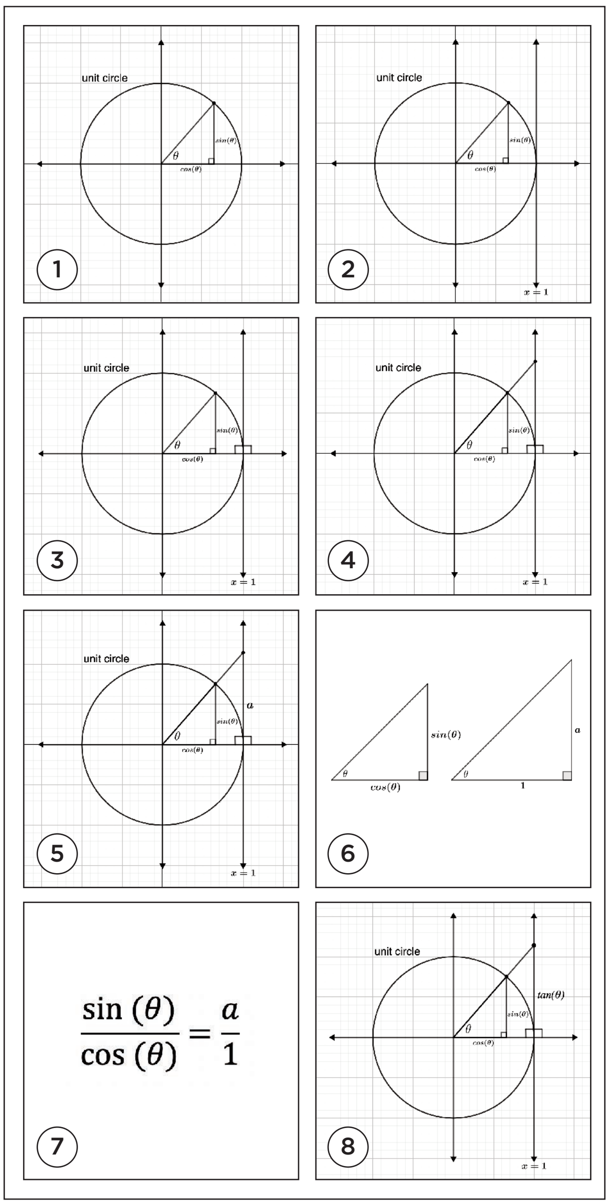

Figure 3. Desired Student Responses.

After summarizing the conclusion of this activity, present students with the diagram below (Figure 4), which shows the visualization of the tangents of select angles.

Students can be asked, "What happens to the tangent values as the number of radians in the angle, θ, increases to $\frac{\pi}{2}$ or decreases to $-\frac{\pi}{2}$?" As θ increases (or decreases), the points along the line x = 1 will also increase in both directions away from (1, 0) until it becomes impossible to draw them. The tangent values are initially easily located on $x = 1$, but as θ approaches $\frac{\pi}{2}$ or $-\frac{\pi}{2}$, the length of the line tends to ∞ or $-\infty$, as represented by the asymptotes on the graph of the tangent function (Figure 1). This again demonstrates why $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ maps to $(-\infty, \infty)$. Hopefully, students will appreciate the elegant connection between this visual and the sizes of the two intervals in question.

As an extension, students can use Figure 4 to make conjectures about the size of [0, 1] as compared to $[1, \infty)$. They can do this by examining the diagram in the following way:

- Look at the tangent values for angles on $\left[0, \frac{\pi}{4}\right]$. They map to [0, 1].
- Furthermore, the tangent values on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ map to $[1, \infty)$.

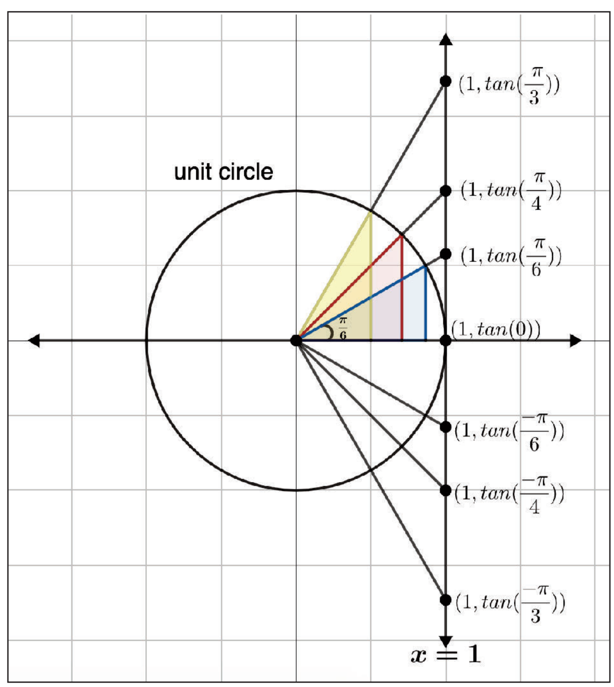

Figure 4. The visualization of the tangents of select angles.

- Since $\left[0, \frac{\pi}{4}\right]$ and $\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ have the same length, and the tangent function is strictly increasing on $[0, \frac{\pi}{2})$, what can be said about the cardinalities of [0, 1] and [1, ∞)?

Hopefully, using this diagram will inspire students to seek proof of their conjectures.

Remarks

Our connection to cardinality is made clear through the progression of studying the tangent function and the concept of matching (and, more mathematically, the idea of a bijection). As we encourage students to consider the sizes of two different sets, we are exposing them to the notion of cardinality as well as Cantor's counterintuitive notions of comparing infinities.

As mathematics educators, we recall how amazed we were when first convinced that [0, 1] and [1, ∞) have the same cardinality. Looking back at Figure 4, clearly the sizes of those "segments" on the line $x = 1$ are vastly different. It is expected that students will develop their own questions and ideas for further study, which has great potential to inspire their enthusiasm for learning mathematics.

For teaching purposes, it would be a good idea to provide an example of two infinite sets with different sizes, so that students do not leave this lesson thinking that all infinite sets have the same size.

For additional extension activities, we recommend tasking students with the following:

1. Compare the following sets of numbers. In each case, decide which set is larger or smaller, or whether they are the same size. If you think the sets are the same size, justify your answer by finding a one-to-one function between the sets.
   a. Integers and even integers
   b. Integers and rational numbers
   c. Natural numbers and whole numbers
   d. Rational numbers and irrational numbers
2. What can be said about the number of points on the real line and the number of points on any segment of that line? Justify your answer.
3. Does the tangent function map to every number on the real number line? Explain why or why not.
4. Explain why the cardinalities of [– 1, 0] and [1, ∞) are equal using the tangent function.
5. Find a bijection that maps [0, 1) to $[0, \frac{\pi}{2})$.
6. Consider the function $f(x) = \tan(x)$ on the restricted domain $(-\frac{\pi}{2}, \frac{\pi}{2})$. This function illustrates a bijection between its domain and range. How could this function be transformed to illustrate a bijection between (0, 1) and (– ∞, ∞)?

For a real challenge, the students can be asked whether the intervals [0, 1] and [0, 1) are the same size. There is a piecewise function that is a bijection between these two intervals, which is an interesting topic for the advanced student to investigate.

Conclusion

In his book *Love and Math* (2014), mathematician Edward Frenkel writes:

> Mathematics is a way to break the barriers of the conventional, an expression of unbounded imagination in the search for truth. Georg Cantor, creator of the theory of infinity, wrote: "The essence of mathematics lies in its freedom." Mathematics teaches us to rigorously analyze reality, study the facts, follow them wherever they lead. (p. 4)

We believe these sentiments should inspire both teachers and students.

The ideas of set theory are accessible to high school students; however, they are almost never taught until college. It is easy to engage students with questions such as, "How many numbers are there between 0 and 1?" and the use of physical models such as the unit circle that they can draw themselves. In fact, previous research has shown that "students use intuitively the same methods [to compare] infinite sets…[and] finite sets. Although students have no special tendency to use 'correct' Cantorian…'one-to-one correspondence,' they are prone to visual cues that highlight the correspondence" (Pehkonen et. al., 2006, p. 346). What is difficult in teaching Common Core Algebra II is finding appropriate places to supplement the curriculum in order to provide enrichment for students and pose interesting and inspirational mathematical questions. A viable option to alleviate this dilemma is to use the tangent function, already in the curriculum, as a launching point to demonstrate ideas about cardinality. This method is visually accessible, rigorous, and innovative. More specifically, it gives high school students seeking enrichment the opportunity to delve into set theory by providing an analogy between the tangent function and notions of infinity.

Acknowledgements

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