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The Secondary-Tertiary Transition in Mathematics: What High School Teachers Do to Prepare Students for Future Success in College-Level Calculus

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ABSTRACT Quantitative analysis of the Factors Influencing College Success in Mathematics (FICSMath) Survey data indicates that high school mathematics teachers' abilities to teach for conceptual understanding is a significant and positive predictor of student performance in single-variable college calculus. To explore these findings further, we gathered and analyzed interview data gained from a representative sample of high school precalculus teachers from across the U.S., identified by their students as requiring high levels of conceptual understanding (n=13). Seventeen themes were identified and then combined into five overarching phenomenological themes. These overarching themes suggest that teachers who teach for high conceptual understanding (a) support relational understanding during problem solving, (b) require students to learn how to study to build on prior knowledge and learn from mistakes, (c) use mathematical language and ask critical questions to support learning, (d) focus on content knowledge necessary to make connections, and (e) use technology to support learning concepts but limit calculator use. Comparison of these results to quantitative findings further illuminate that intentional development of disciplinary knowledge, cognition, and language are noteworthy points of intersection for teachers and researchers alike.

KEYWORDS *conceptual understanding, mathematics, high school teachers, pedagogical practices, precalculus, college calculus*

As low student enrollment and high attrition among college students in science, technology, engineering, or mathematics (STEM) fields challenge STEM education and career growth (Sithole et al., 2017), the need to investigate what helps students successfully transition from high school mathematics to single variable college calculus is critical. For many college students, college level calculus functions as a crucial gatekeeper in the STEM fields as well as other fields that draw upon mathematics. Bressoud, Mesa, & Rasmussen (2015) confirm: This course requirement often proves to be an "insurmountable obstacle or—more subtly—a great discourager from the pursuit of fields that build upon the insights of mathematics" (p. v).

The many challenges that high school mathematics teachers face as they prepare their students for future success in college level mathematics has led researchers to study the school-to-college transition, often referred to as the secondary-tertiary transition. This transition begins the junior year in high school and extends across the first two years of college (Wade, Sonnert, Sadler, Hazari & Watson, 2016; Gueudet, 2008). The extent and quality of high school preparation during the secondary-tertiary transition in mathematics has been shown to contribute to students' decisions to persist in or move out of STEM majors once they are in college (Wade, Sonnert, Wilkens, & Sadler, 2017; Ellis, Fosdick & Rasmussen, 2016).

While a handful of studies identify possible causes and detail challenges specific to the transition from high school mathematics to college level calculus (Clark & Lovric, 2008; Selden & Selden, 2001), few examine what can be done to effectively prepare high school students for success in college level calculus. A growing body of research identifies teaching for high conceptual understanding as a significant and positive predictor of future performance in tertiary calculus (Wade, Sonnert, Sadler, & Hazari, 2017). This study draws on this research and investigates what high school precalculus and calculus teachers—identified as teaching for high conceptual understanding—do to prepare students for future success in college level calculus

The FICSMath Project

The Factors Influencing College Success in Mathematics (FICSMath) Project remains the largest and most recent national study of the secondary-tertiary transition. In the fall semester of 2009, college freshmen enrolled in single variable college calculus at public and private colleges and universities across the U.S. completed the FICSMath Survey. A total of 10,492 surveys were collected from 134 two- and four-year post-secondary institutions. On this survey, students responded to questions about their educational experiences in their last high school mathematics course. Professors secured students' completed surveys until the end of the semester. Before returning the surveys to the Harvard-Smithsonian Center for Astrophysics, professors recorded final grades for each student on the student's respective survey. The collection of this data allowed the relationship between students' high school instructional experiences and actual performance in college-level calculus to be examined.

Quantitative analysis of this data found the conceptual understanding survey item to be a significant predictor of future performance in tertiary calculus (Wade, Sonnert, Sadler, & Hazari, 2017). That is, among respondents who (a) took secondary precalculus or calculus courses in the spring of their senior year in high school, and (b) took tertiary calculus in the fall semester, those who reported experiencing high school mathematics instruction requiring high conceptual understanding performed better in their tertiary calculus course. This research advances understanding of the secondary-tertiary transition from the students' perspective. What remains unknown, however, is the relationship between what students reportedly experienced and what their high school teachers actually did to teach for high conceptual understanding in their classrooms.

Our interest in understanding what high school teachers do to successfully prepare secondary students for future success in tertiary calculus led us to contact teachers identified as teaching for high conceptual understanding on the FICSMath Survey. Because the FICSMath Survey asked respondents to provide their most recent high school mathematics teachers' names, we were able to interview a systematically-selected national sample of precalculus and calculus teachers. These teachers described how they defined and promoted conceptual understanding in their classrooms as well as what they did to prepare students for tertiary calculus success.

Investigating Conceptual Understanding

Conceptual understanding has been a prominent topic in mathematics education research for many years. For example, Brownell (1935), Davis (1984), Hiebert & Carpenter (1992), and Hiebert & Grouws (2007) define conceptual understanding as making mental connections among mathematical facts, procedures, and ideas. A dominant assumption in this research is that the end goal for conceptual understanding is knowledge that can be used to "recognize, identify, explain, evaluate, judge, create, invent, compare, and choose" (Star, 2000, p. 3). Skemp (2006) extends this research by describing two different types of understanding that occur in school mathematics: relational and instrumental understanding. Relational understanding implies that students know what to do and why, whereas instrumental understanding indicates that students know rules without reason. Another way to consider relational understanding is through Russell's (1999) web of mathematical memory. This web networks ideas and concepts in long-term memory. Mathematical memory is then retrieved from working memory to support meaningful mathematical problem solving.

This research illuminates the critical role that conceptual understanding plays in problem solving and the learning of mathematics. Quantitative analysis of the FICSMath Survey concludes similarly: Conceptual understanding proved to be a significant and positive predictor of performance in tertiary calculus (Wade, Sonnert, Sadler, & Hazari 2017). In Wade et al.'s (2017) in-depth correlational study of the 70 instructional items on the FICSMath Survey, four constructs correlated with the conceptual understanding survey item: multiple representations (5 items; $r=0.301$), applications (4 items; $r=0.207$), discussion (4 items; $r=0.254$), and mathematical fluency (5 items; $r=0.402$). Use of a hierarchical linear model (HLM) further revealed the mathematical fluency construct to be a positive predictor of future performance in tertiary calculus. (For a full report of these findings, see Wade, Sonnert, Sadler, & Hazari, 2017). Intrigued with these findings, this study allowed us to more richly explore what high school teachers—identified by former students on the FICSMath Survey as teaching for high conceptual understanding—do to prepare students for future success in college level calculus.

Research Questions and Study Design

This study is a follow-up qualitative investigation of the FICSMath quantitative results linked to conceptual understanding. In this phase, we explore what teaching for conceptual understanding means and involves for a national sample of high school teachers identified on the FICSMath Survey as teaching for high conceptual understanding. That is, on a 0-5 scale, where 0-1 indicated a low level of teaching for conceptual understanding, and 2-3 indicated a medium level, these teachers were rated at a high level (4-5) of teaching for conceptual understanding.

We base our approach to qualitative inquiry in phenomenology. Like all qualitative research, this approach is concerned with exploring phenomena "from the interior…taking the perspectives and accounts of research participants as a starting point" (Ritchie, Lewis, Nichols, & Ormston, 2014, p. 3). What is particular to phenomenology, however, is its focus on how a group of individuals makes sense of the same phenomenon based on their experience and the meaning that it holds for them (Akerlind, 2005). A major advantage to using this approach, then, is that it allowed us to unpack conceptual understanding as the teachers defined it in their own terms and enacted in their own ways.

In theorizing about understanding (conceptual or otherwise), Gadamer (1996) suggests that understanding is generated in the "fusion" and interplay among various perspectives or "horizons" (p. 306). Rather than objectify conceptual understanding as a phenomenon that these teachers alone can fully explicate, we endeavored to discover the spaces where multiple perspectives confirm, challenge, complicate, and expand each other. Although our deliberation of conceptual understanding in this paper focuses on the teachers' perspectives, we include our own perspectives and perspectives prevalent in the literature as well. We do so not as a matter of testing teachers' knowledge in relation to what we or others know about conceptual understanding, but to honor their voices and situate their contributions (which correlates both with students' notions of conceptual understanding and students' subsequent performances in tertiary calculus) within the larger conversation about conceptual understanding in ways that help us to better prepare high school students for later success in college-level mathematics. Accordingly, the following questions frame this study:

1. How do these high school teachers define conceptual understanding in mathematics?
2. What do they say that they do to promote conceptual understanding in the precalculus and/or calculus courses they teach?
3. What do they believe are the best ways to prepare students for college calculus success?
4. How do the findings generated from this study inform the field about conceptual understanding and the secondary-tertiary transition?

Regarding the last question, we relate what we learned about conceptual understanding through this study to what we learned through quantitative research previously conducted. In so doing, we provide a richer, collective understanding of what teaching for conceptual understanding means.

Method

The FICSMath Teacher Sample

Of the 10,492 students who completed the FICSMath Survey, a subset of 2,326 students had precalculus their senior year in high school. Within this subset, 1,285 (55%) students indicated that their precalculus teacher required high conceptual understanding. Likewise, there were 4,912 respondents who reported they had some level of calculus their senior year in high school. From these, 3,248 (66%) reported their teachers required high conceptual understanding. The FICSMath Survey respondents had the option of reporting their senior level high school mathematics teacher's name and the name of their high school attended. If provided, this data was used to identify teachers for this study.

From those teachers who met the aforementioned criteria and for whom we had names and contact information, we randomly selected and emailed a sample of teachers from each of the four U.S. Census Regions an invitation to participate in the study. Twenty-six teachers from the West, 24 from the Midwest, 17 from the South, and 17 from the Northeast were contacted. Of the 84 teachers contacted, 13 (8 females; 5 males) agreed to be interviewed. Table 1 lists the number of teachers we interviewed by U.S. Census Region. The number of years of teaching experience among participants per region averaged twenty years or more (West: 22 years; Midwest: 23 years, South: 25 years, and Northeast: 20 years respectively). Thus, the pool of teachers interviewed can be described as experienced and tenured.

Table 1

Number of Teachers Interviewed by U.S. Census Region and Gender

US Census Region	Males Interviewed ($n=5$)	Females Interviewed ($n=8$)	Total Teachers Interviewed ($n=13$)
West	2	1	3
Midwest	3	2	5
Northeast	0	2	2
South	0	3	3

Data Collection

In-depth interviews were a key component of this study. Our semi-structured, open-ended interview protocol focused on the first three research questions and also collected information about the secondary mathematics courses and grade level(s) they taught. Because the teachers were geographically dispersed across the U.S. and face-to-face interviews were not possible, we conducted these interviews by phone. Interviews averaged 30 minutes. All were recorded and transcribed for the purpose of analysis.

Phases of Analysis

Our approach to this research relied on the gathering and analysis of phenomenological interpretations and experiences. Therefore, interpretative phenomenological analysis guided how we examined individual teacher responses to the first three research questions (Saldaña, 2015). The analysis occurred in four phases.

First Phase

To honor the teachers' voices and ground the analysis in their own perspectives, we used in vivo coding, open coding, and analytic memos in the first cycle of analysis (Saldaña, 2015). Interview data were aggregated by research questions. The researchers read teacher responses to each question with the purpose of identifying (a) significant teacher statements that distilled, summarized, or added value to each of the three questions posed, (b) patterns and regularity of interpretation and action, and (c) any statements perceived as idiosyncratic or worth thinking about further. We also wrote analytic memos to provide a visible 'audit trail' as we moved from 'raw' data through preliminary coding to final codes.

Second Phase

Members of the research team met at significant points throughout the data collection process to share quotes, identify patterns and regularity, discuss emerging codes across the 13 interview excerpts, and reflect on idiosyncrasies that challenged or added to our thinking. In Table 2, we present a sampling of quotations for one of the interview questions to show how we (a) examined teacher statements looking for repeated words or ideas to generate preliminary codes, and (b) arrived at final thematic codes. The same process was used for coding all of the three interview questions.

Third Phase

As themes emerged and recurred in the data, we shifted from an exploratory to a confirmatory stance and addressed issues of validity. According to Dahlin (1999), the validity of phenomenological research is based on three factors. The first factor is that major themes must be independent. Once codes were finalized, we conducted a check to determine the independence of the themes assigned to teacher statements. Inter-rater reliability (i.e., the extent to which two raters agreed on the assigning of teacher statements to themes) was established through comparing coding across two raters. Cohen's Kappa was computed to be 0.70. The adjusted agreement (accounting for chance agreement) between the two raters was 70%, thereby indicating good rating consistency between raters (Landis & Koch, 1977).

The second factor is plausibility. Plausibility requires that the categories represent actual or possible experiences (Dahlin, 1999). To ensure plausibility, codes and categories were derived from the teachers' actual words rather than pre-defined categories. Members of the research team also checked the generation of codes and categories for consistency of interpretation.

The last factor important to validity is correspondence between the generated themes and what is known from previous studies of the phenomenon. As mentioned previously, this study is a qualitative investigation of the FICSMath quantitative results linked to conceptual understanding. Thus, we address the need for interconnectedness of data by comparing the results gained from this study to quantitative findings gained from an earlier study (Wade, Sonnert, Sadler, & Hazari, 2017).

Table 2

Sample of Initial Coding Categories for Interview Question: What Teachers do to Teach for Conceptual Understanding

<table>
<tr><th>Raw Data</th><th>Preliminary Code</th><th>Final Thematic Code</th></tr>
<tr><td>A lot of students think “If I can get the right answer, then I know math.” But that is not what math is about. I tell my students to look at the concepts. Think strategy, think ideas. It is about the steps and the process, not the answer itself.</td><td>Think concepts, strategy, and process, not only about the final answer.</td><td rowspan="3">Connect problem solving concepts to prior knowledge and allow students to struggle.</td></tr>
<tr><td>Just doing the problem is not an effective way to teach understanding. Let them [students] present ideas and struggle</td><td>Allow students to struggle through the problem-solving process.</td></tr>
<tr><td>I try to give them a framework they can hang their procedures on.</td><td>When problem solving, hook to prior knowledge.</td></tr>
<tr><td>I spend a lot of time trying to understand the details of the material and be able to see it from different perspectives. Then I digest the important points into the notes I give them, and outline the important aspects I want them to get out of it [the lesson].</td><td>Teachers utilized their own content knowledge to bring emphasis to the concepts.</td><td rowspan="2">Apply content knowledge.</td></tr>
<tr><td>When a student asks me a question that I know I have answered before, I have been able to look at it differently, state it a different way, or use some other example that they already know and show how it relates.</td><td>Teachers content knowledge allows them to think flexibly about the content they teach.</td></tr>
</table>

Final Phase

To answer our fourth research question, we used axial coding to deliberate on the relationships among all themes within and across the first three research questions (Saldaña, 2015; Spiggle, 1994). This layer of analysis allowed us to identify overarching phenomenological themes. We then compared these overarching themes (from the teachers' perspectives) to the FICSMath Study quantitative results (from the students' perspectives).

Researcher Reflexivity

Our involvement in this project stems from a shared background and interest in secondary mathematics education and teacher education. As past and present high school teachers and researchers ourselves, we wish to counter the trend of students leaving STEM fields because of difficulty with the secondary-tertiary transition in mathematics. This study allowed us to address a main limitation of the quantitative FICSMath Study, which follows: When it comes to teaching for high conceptual understanding, what do these high school teachers say they do to prepare their students for future success in tertiary calculus?

Findings

The study's research questions frame the sharing of themes gained from our analysis. Within each question, we discuss each theme, moving from themes that have the greatest to the least number of teacher responses. A total of 17 themes emerged from this analysis—four, eight, and five themes respectively. To answer the fourth research question, we present five, overarching phenomenological themes gained through the deliberation on all themes identified within and across the first three research questions.

Definition of Conceptual Understanding

Teacher responses to the question, “What is the meaning of conceptual understanding in mathematics?” generated 20 statements that resulted in four themes: (a) applications are demonstrated (6 statements), (b) relational understanding is displayed during problem solving (5 statements), (c) learning is discussed (5 statements), and (d) students build on prior knowledge (4 statements).

Applications Are Demonstrated. Teachers defined conceptual understanding as students being able to apply mathematical concepts in novel or different contexts. As one teacher shared, "students are able to apply what they learn to real-life situations." Another teacher clarified, "A student who has a good understanding of a concept should be able to apply that concept in a lot of different circumstances, and a lot of problems that look different but are rooted in similar concepts." Lastly, one teacher expressed, "They [students] would be able to see something – a problem or situation – they have never seen before and be able to use the tools they have learned to approach it and solve it." Interestingly, application problems in mathematics are often used to improve student engagement in learning mathematics (Beswick, 2010). These teachers suggest that application problems serve other purposes beyond student engagement.

Relational Understanding is Displayed During Problem Solving

Here, teachers identified the importance of sense making during the problem-solving process. As one teacher stated, "It's not [students thinking about] *How do I do it?* It's *What am I doing?*" Another explained, "Understanding is demonstrated when students bring up a topic from several weeks ago and are still able to understand and explain it." These views echo what Skemp (2006) identifies as relational understanding.

Learning is Discussed

For these teachers, mathematical discourse is a tool used in class to share and support understanding. As one teacher clarified, "A lot of times you can get to conceptual understanding through discussions." The asking of open-ended questions such as, "What does it mean when a binomial is in parenthesis?" were also deemed important. They also value students asking each other questions, indicating that "conceptual understanding happens during work time when students are asking each other questions." These statements speak to the importance of mathematical discourse, which involves students engaging in meaningful problem solving, while conjecturing, scrutinizing, and defending problem solving ideas (Lampert, 1989; Ball, 1993).

Students Build on Prior Knowledge

Teachers expressed the importance of mathematics building upon itself and stressed wanting to help students see and forge connections across content and courses. One teacher shared, "Everything with math builds. You have to have that foundation." Another stated, "[They should] understand the mathematical principles that lead to the procedures, as opposed to memorizing the procedures." Another teacher emphasized the need for helping students gain a "firm grasp on what the essence of something is." Yet another said, "Being able to make connections, see those connections, and put them into a new situation so that you don't just understand the fact, but are able to pull the idea forward and build upon several other ideas." These ideas support mathematical principles and concepts being stored in long-term memory and being available for recall to support the learning of more new mathematics (Sweller, van Merrienboer, & Paas, 1998).

What Teachers do to Teach for Conceptual Understanding

Teacher responses to the question, "Describe how you have helped students understand concepts in mathematics" generated 57 statements that coalesced around eight themes: (a) connect problem solving to prior knowledge and allow students to struggle (12 statements), (b) assess student work (9 statements), (c) provide opportunities for students to learn from their mistakes (6 statements), (d) apply content knowledge (5 statements) , (e) focus on the language of mathematics (five statements), (f) use representations (5 statements), (g) use technology to support concepts (3 statements), and (h) ask critical questions (2 statements).

Connect Problem Solving Concepts to Prior Knowledge and Allow Students to Struggle.

The number of teacher statements in this theme is the greatest across the first three research questions. Teachers shared various ways they support students as they build on prior learning and build new mathematical concepts. One stressed, "After hooking to prior learning, I let them figure out problems based on what they already know." Two teachers highlighted the importance of the problem-solving process as compared to the final answer. Exemplifying this, one stated, "A lot of students think, 'If I can get the right answer, then I know math,' but that is not what math is about. I tell my students to look at the concepts. Think strategy, make connections across ideas. It is about the steps and the process, not just the answer itself." The other teacher added, "The questions on tests are never straightforward." Another shared, "the structure of the classroom – sitting together, not reviewing, establishing their own habits and asking themselves questions" as ways to support understanding. Furthermore, addressing the willingness to struggle through the problem-solving process, one teacher stressed, "They [students] have to have done something mathematical towards solving the problem and they need to learn to get past the fear that they have." When teachers incorporate such opportunities for students to develop conceptual understanding, successful problem solving can occur (Haskell, 2001). The term "problem solve" has multiple meanings, but it is most often associated with solving nonstandard problems (Darken, Wynegar, & Kuhn, 2000). Such problems require students to “grapple with new and unfamiliar tasks when the relevant solution methods are not known” (Schoenfeld, 1992, p. 56).

Assess Student Work

Teachers saw assessment of student work as a way to support their understanding. For example, one teacher stated, “I don’t accept a blank problem with a question mark on homework.” Another responded, “They [students] receive a zero if they leave a blank answer because they have to learn to attempt every problem that they see.” Other teachers stressed the types of assessments they grade. For example, one teacher shared, “More of the grading is on tests and quizzes where they have to demonstrate knowledge versus on homework.” Schoenfeld (2016) refers to assessing problem solving as providing access to what students know and can do from their prior learning.

Provide opportunities for students to learn from their mistakes

Teachers valued students being persistent problem solvers. As one teacher explained, “I try to teach them [students] it is okay to get the wrong answer when they’re learning because it isn’t always about getting the right answer.” Another shared, “It is important that students learn from their and others' mistakes.” Additionally, one teacher indicated, “When we go over a problem together, I’ll ask them if they [the students] thought of certain points and ask them why the answer is not something or why an incorrect answer is unreasonable.” When it comes to teaching AP Calculus, one teacher noted, “Giving students old AP questions and asking why certain answer choices are wrong supports the reasoning process by analyzing wrong answers and working backwards.” These practices align with the idea of using flawed reasoning as being a part of the problem-solving process. When students try a problem-solving method that does not work, they then have the opportunity to learn from flawed reasoning (Russell, 1999).

Apply Content Knowledge

Teachers shared how they use their own content knowledge when preparing to teach mathematical concepts. One teacher indicated, “If you’re going to teach high school mathematics, you’re going to have to take a bit [of mathematics] in college.” Another added, “I spend a lot of time trying to understand the details of the material and see it from different perspectives. Then I digest the important points into the notes I give them and outline the important aspects I want them to get out of it [the lesson].” Another stressed, “My examples are very specific. When you’re creating an example, if you haven’t thought each step through, and then you get in the middle of it and think ‘Oh gosh, I didn’t think it would go down this path,’ then that’s not a good plan.” In a similar vein, one confirmed, “the underlying principles—it’s kind of theoretical” as being essential focal points when teaching mathematics. Additionally, one teacher shared, “when a student asks me a question that I know I have answered before, I look at it differently, state it a different way, or use some other example that they already know and show how it relates.” These teachers echo the importance of teachers’ content knowledge for pedagogical purposes, as demonstrated by a rich literary base (Ball, Thames, & Phelps, 2008; Krauss et al., 2008; Skemp, 2006).

Focus on the Language of Mathematics

These teachers support students' understanding of the language of mathematics. One teacher stated that she encourages students to “not just regurgitate a definition but describe something in their own words and write their own problem that utilizes the idea.” Another teacher offered, “I try to get them [students] to understand a definition or a concept in as few words as possible.” Another teacher simply said, “I emphasize definitions a lot.” When it comes to mathematical literacy, it is critical that students use precise mathematical language, syntax, and symbolism in oral and written work (Wade, Sonnert, Sadler, Hazari, 2017).

Use Representations

Here, mathematics teachers shared their view that representations, such as “manipulatives, everyday examples, and visualizations,” are used to help students understand mathematical concepts. Teachers also discussed, “Connecting concepts to visual diagrams” to support students making connections across concepts. Modeling mathematics was also shared as a way to “explain confusions.” Calculus teachers discussed representing ideas “geometrically and taking things apart piece by piece.” Likewise, teachers who also taught Algebra I referred to representations as a way to “explain balance and equivalence.” These ideas align with Tall’s (1997) view of representation. He considers representations as ways to demonstrate how functions “do and undo” and explain foundational, yet complex, mathematical relationships and ideas (p. 7).

Use Technology to Support Concepts

Teachers shared how they use technologies to support understanding. For example, one teacher stated "I use graphing calculators in the classroom, which helps them [students] to look at things from different perspectives." Another teacher indicated, "I usually spend at least 3-4 days in the computer lab with them [students], working on activities that construct what the derivative is through different approaches. Getting them [students] to actually play with these things themselves and having to think on their own is good for them, especially in Calculus when the concepts are challenging." Lastly, one teacher mentioned, "I found Geometer's Sketchpad to be useful for teaching calculus and precalculus." In a similar vein, the National Council of Teachers of Mathematics (NCTM) recommends that teachers and students have regular access to technology to support mathematical sense making, reasoning, problem solving, and communication (NCTM, 2015).

Ask Critical Questions

These teachers reported asking students critical questions, such as "Can you come up with an equation based on what you read? Can you understand what you're looking at? Do you understand why you are doing the steps you're doing?" Another shared, "I try to give them [students] problems that force them to think about the actual concepts, something more than just 'find the derivative of f(x),' but instead, I ask questions, such as 'What does the derivative mean?'" This theme aligns with the view of Krauss et al. (2008) that one way teachers can push students' thinking to higher levels is through asking critical questions.

Teaching for College Calculus Success

Lastly, we asked the teachers, "What do you believe is the best way to prepare students for college calculus success?" Teacher responses generated 24 statements that fell within five themes: (a) focus on foundational content (10 statements), (b) require conceptual understanding 5 statements), (c) require students to learn how to study (4 statements), (d) focus on the language of mathematics (3 statements), and (e) limit graphing calculator use (2 statements).

Focus on Foundational Content

Teachers expressed the need to develop foundational content knowledge. One expressed, "They [students] need to have a good understanding of algebra. The students have as much trouble with calculus as they do with algebra, especially graphing." Addressing the need to spend more time on algebra, one teacher shared, "Students need to feel confident and competent in algebra. Rushing through those courses does a disservice to students when they reach upper level classes." Additionally, one reflected, "Algebra II and pre-calculus are critical for calculus preparation. If you are missing any of those pieces, it's a real problem." Other teachers addressed specific content: "Logarithms should be a focus because they [students] still do not fully understand exponents. The properties are strange, and in Algebra II the domain issues are ignored. So, when you try to introduce that, they [students] reject it." One teacher also shared, "I focused more on the symbols, the functions and their properties." Wade, Sonnert, Sadler, Hazari, and Watson (2016) examined similarities and differences between teachers' and professors' perspectives on preparing students for college calculus success. The professors mentioned algebra, more than anything else, as foundational preparation for college calculus success. These teachers' views echo the professors' perspectives of the importance of learning algebra.

Require Conceptual Understanding

These teachers required students to understand mathematical concepts. "They [students] are not always sure why they're doing what they're doing, one teacher noted. "Instead of just giving the rules, they have to have some part in discovering the rules." Another one remarked, "I spend a lot of time on the "why." Why do we use this method? Why is it better than the other method? If they know how and why they are doing what they're doing, then they will be successful in advanced classes." In terms of understanding specific content, another teacher responded, "Conceptual understanding of graphs and functions are the basic building blocks for calculus. Working with graphs and functions in a way that requires deeper conceptual understanding trips up most students." These views of how to require conceptual understanding align with knowing what to do and why, which aligns with Skemp's (2006) definition of relational understanding.

Require Students to Learn How to Study

Teachers indicated that learning mathematics requires intentional focus that comes from studying. One teacher indicated, "Give students lots of suggestions for how to study. Students tend to get to precalculus and do not do as well as they previously had. They expect the information to simply come to them rather than to pursue the ideas." Another participant stated, "A lot of students get to calculus without having to work or study in math class. They are not used to having questions on quizzes or tests that they have not seen on their homework. So, it is important to get them used to that before they get to calculus." Clark and Lovric (2008) identify being able to study as one of the important shifts students must make as they transition from lower to higher level mathematics.

Focus on Mathematical Language

Teachers identified the need to use disciplinary mathematical language correctly. One teacher explained "We do them [students] a tremendous disservice when teachers use nonsense terminology such as, 'for x+2=7 move the 2 to the other side,' rather than teaching using actual operations involved." Speaking more broadly about language, another teacher stated, "Algebra II is the language of preparing students for college calculus." Tall (2004) identifies mathematical language as essential towards connecting mathematical ideas, perceptions, and concepts necessary to develop abstract thought.

Limit Graphing Calculator Use

Teachers addressed that the AP Calculus exam requires the use of calculators. One teacher clarified, "Students don't use a calculator in calculus classes until the very end of the year (February). A graphing calculator can be used to illustrate concepts to the class such as how a graph of a derivative is affected by something. However, students should create everything by hand so they can do everything with their brains and appreciate the technology." This idea aligns with Wade, Sonnert, Sadler, and Hazari (2017) and the NCTM's (2015) views regarding the strategic use of technology. Both groups recommend that calculators are used to support concepts, but not to replace problem solving efforts.

Overarching Phenomenological Themes

Table 3 addresses our fourth research question: How do the findings generated from this study inform the field about conceptual understanding and the secondary-tertiary transition? To address this question, we used axial

Table 3

Alignment Across the Themes Generated from the Three Interview Questions

Overarching Phenomenological Themes	What does it mean to conceptually understand mathematics?	How you have helped students understand concepts in mathematics?	What is the best way to prepare students for college calculus success?
Theme 1: Support relational understanding during problem solving.	Relational understanding is displayed during problem solving.	Connect prior knowledge to problem solving concepts and allow students to struggle.	Require conceptual understanding.
Theme 2: Require students to learn how to study so they can build on prior knowledge and learn from their mistakes.	Students build on prior knowledge.	Provide opportunities for student to learn from their mistakes. Assess student work.	Require students to learn to study
Theme 3: Use mathematical language and ask critical questions to support learning.	Learning is discussed.	Focus on language of mathematics. Ask critical questions.	Focus on mathematical language.
Theme 4: Teachers focus on their own content knowledge necessary to make connections across mathematical applications.	Applications are demonstrated.	Apply content knowledge. Use representations.	Focus on content.
Theme 5: Use technology to support learning concepts but limit calculator use.		Use technology to support concepts.	Limit graphing calculator use.

coding. Axial coding allows interrelationships to form and provides the basis for informing a field and theory construction (Saldaña, 2015; Spiggle, 1994). We arrived at these themes by specifying relationships and delineating core themes around which all 17 themes revolved (Goulding, 2005). Five overarching phenomenological themes emerged from this deliberation. All told, these overarching themes suggest that teaching for conceptual understanding should (a) support relational understanding during problem solving, (b) require students to learn how to study to build on prior knowledge and learn from mistakes, (c) use mathematical language and ask critical questions to support learning, (d) focus on content knowledge necessary to make connections, and (e) use technology to support learning concepts but limit calculator use. These overarching themes, shown in Table 3, advance current knowledge of precalculus and calculus instructional practices in relation to conceptual understanding and of their perceived benefits across the secondary-tertiary transition.

Discussion

We now consider the relationship between the overarching phenomenological themes gained from this qualitative study (from the teachers' perspectives) and quantitative results gained previously from the FICSMath Study (from the students' perspective). Table 4 (next page) shows our thinking about how these constructs and the phenomenological themes relate. It is important to note that Theme 2 (Require students to learn to study so they can build on prior knowledge and learn from their mistakes) and Theme 5 (Use technology to support learning concepts but limit calculator use) are not included in the table because they did not align with the constructs from the quantitative study.

In Table 4, we identify the three overarching phenomenological themes in relation to the four constructs correlated with the conceptual understanding item on the FICSMath Survey (multiple representations, applications, discussions, and mathematical fluency respectively). Of the four quantitative constructs listed, mathematical fluency is the only construct that was a positive predictor of performance in tertiary calculus.

Interestingly, the focus on mathematical language emerged as a theme for two of our research questions (RQ2 and RQ3). This finding is not to be taken lightly. As Kenney (2005) points out, "Mathematics is truly a foreign language for most students," thus making its acquisition "an extremely difficult process" (p. 3). Reasons for this phenomenon are many with the greatest difficulty stemming from the "double decoding" that occurs during the entire process:

Double decoding…occurs when we first encounter written mathematics or symbols, which must first be decoded, and then connected to a concept that may or may not be present in prior knowledge even in an elementary way" (p. 5)

Wakefield (2000) suggests that mathematical language requires memorization of symbols, algorithms, and abstractions that improves over time with practice. Furthermore, Wakefield posits that the meaning of mathematical language is influenced by symbol order (or syntax) and that understanding requires both decoding and encoding. In sum, these teachers suggest that they provided explicit instruction around mathematical language so that students gained mathematical fluency that supported their understanding of mathematical concepts at a deeper level. Teaching for high conceptual understanding, therefore, requires intentional development of disciplinary knowledge, cognition, and language.

Limitations of the Study and Future Research

The FICSMath quantitative study pointed to conceptual understanding as needing additional research, and thus, this follow-up qualitative study was designed. The focus sought to provide a more in-depth knowledge of conceptual understanding as it relates to what teachers say they do to support understanding across the secondary-tertiary transition. While 84 teachers were contacted who were reported as teaching for high conceptual understanding on the FICSMath Study, only 13 agreed to participate. This is a small number, and the low proportion of participants raises the concern of potential non-response bias. On the other hand, our sample has the strength of not being a convenience or snowball sample but is based on a systematic sampling frame. This, of course, made data collection much more arduous. The work that it took in contacting and interviewing this number of teachers spanned two years. Significant time and effort were invested to connect two data sets (the FICSMath Study and the contact information of teachers and schools) to assure teachers we reached out to were indeed identified by their prior precalculus or calculus students as teaching for high conceptual understanding. Additionally, we sought to have an equal balance between female and male teachers, but as Figure 1 shows, five male and eight female teachers participated in this study. Out of the 53 teachers contacted, 30 were males, yet fewer males responded to our request to participate in an interview.

Table 4

Relationships Across Overarching Phenomenological Themes (Qualitative Study, Teachers' Perspectives) and Conceptual Understanding Constructs (Quantitative Study, Students' Perspectives)

Qualitative Theme	FICSMath Quantitative Constructs	FICSMath Survey Items in Constructs	Significance
Theme 1: Support relational understanding during problem solving.	Multiple Representation	• Highlighted more than one way of solving a problem • Explained ideas clearly • Used graphs, tables, and other illustrations • Presented various methods for solving problems • Teacher was enthusiastic about mathematics	"Multiple representation" was a positive predictor of future performance in tertiary calculus for secondary precalculus students, but not for calculus students. This may point to AP exam preparation that is focused on test taking strategies rather than support for relational understanding. These findings align with Bressoud's (2015) work. How to prepare students for AP calculus with depth of understanding, however, remains critical to investigate.
Theme 3: Use mathematical language and ask critical questions to support learning.	Mathematical Fluency	• Emphasis on precise definitions • Emphasis on vocabulary • Emphasis on hands-on activities/labs • Emphasis on functions • Emphasis on mathematical reasoning	The importance of mathematical language (definitions, vocabulary, functions, which include syntax and symbolism) is confirmed by the mathematical fluency construct being a positive predictor of performance in tertiary calculus.
	Discuss	• Students questions and comments were valued • Class discussions were useful • Student were comfortable asking questions • Teachers answers were valuable	While mathematical discussions align with Theme 3, the discussions construct was not a significant predictor of performance in tertiary calculus. Use of mathematical discourse to increase student performance remains a critical issue to examine.
Theme 4 Focus on teachers' own content knowledge necessary to make connections across mathematical applications.	Applications	• Connected math to other subject areas • Connected math to real life applications • Connected math to everyday life • Examples from everyday world were used.	While teachers' content knowledge is critical, using it to make connections across mathematical applications was beneficial for top-performing students in secondary calculus, but not for lower-performing students. How to connect application problems to support learning for all students also requires further study.

Possible areas of additional research—illuminated by the quantitative and qualitative studies—are as follows: (a) how to prepare secondary calculus students for the AP calculus exam, and for tertiary calculus, with depth of understanding while covering the breadth of material; (b) how to connect mathematical discourse to increase student performance; and (c) how to connect application problems in secondary precalculus and calculus courses to support the learning for all students; Additionally, the two overarching phenomenological themes that did not relate with the quantitative study would also benefit from more research. It remains unclear how Theme 2 (Require students to learn to study so they can build on prior knowledge and learn from their mistakes) and Theme 5 (Use technology to support learning concepts but limit calculator use) fit in the secondary tertiary transition. More research in all of these areas may shed light on the secondary-tertiary transition. All told, the goal remains to address the attrition among college students in STEM fields by improving students' preparation for the secondary-tertiary transition.

Conclusion

Learning, as defined by cognitive load theorists, is a permanent change in long-term memory, meaning that what is learned can be recalled and applied to support the processing of more new information (Sweller, Van Merriënboer & Paas, 1998). When teachers are teaching for high conceptual understanding, they are not only teaching for current but future learning as well. Thus, the challenge of teaching during the secondary-tertiary transition is not an easy task.

As mentioned in the beginning of this paper, much depends on the robustness of the instruction that mathematics teachers provide at the high school level (Clark & Lovric, 2009). Given the critical role that conceptual understanding plays in preparing high school students for future success in college calculus, problem solving, and the learning of mathematics, understanding what teachers do to effectively achieve it is of great value. With that in mind, we offer instructive ideas on which teachers and researchers alike can act.

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