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Opportunities in Mathematics Content Courses for Developing Prospective Teachers' Knowledge About Students

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ABSTRACT I qualitatively reviewed mathematics education literature to offer explicit classroom-based examples and recommendations on the learning opportunities that can be embedded into mathematics content courses, designed specifically for prospective elementary teachers (ePTs), to help develop their knowledge of students and their mathematical thinking, learning, and development. I particularly drew on the articles that outlined the course opportunities that could be embedded by college instructors with little or no prior experience teaching or working with elementary students as well as the articles that did not require direct access to schoolchildren in the course. I provided direct quotes and explicit descriptions as representative classroom examples, documented in the literature, for university faculty to select from and plan for implementation in their own courses. Ultimately, in this review of literature, I argue that university faculty who teach mathematics to ePTs, including those who do not have experiences in K-12 educational settings, have numerous readily available resources and classroom-based examples (from the field) to help them embed course opportunities that focus ePTs' attention and reflection (directly and indirectly) on students' mathematical thinking and learning. I also provide research evidence demonstrating that these types of course opportunities help to improve ePTs' readiness to teach and preparedness to work with schoolchildren, particularly in developing their appreciation for students' mathematical successes and challenges.

KEYWORDS *mathematics content courses; prospective elementary teachers; knowledge of students*

In 2001, the Conference Board of Mathematical Sciences (CBMS) reported evidence of too many prospective elementary teachers (ePTs) graduating college with insufficient knowledge of mathematics education for the effective teaching of children (CBMS, 2001; also see Ball, Sleep, Boerst, & Bass, 2009; Greenberg & Walsh 2008). As a result, CBMS proposed that all institutions preparing elementary teachers must offer and require at least nine credits of mathematics *content* courses that are designed specifically to better prepare ePTs to teach mathematics in accordance with the vision that “teaching elementary mathematics requires both a wide range of pedagogical

skills and considerable mathematical knowledge” (CBMS, 2012, p. 55).

In the US, nearly all (90%) mathematics content courses, designed for prospective elementary teachers, are taught and developed by the mathematics department faculty and staff (Masingila, Olanoff, & Kwaka, 2012). A variety of individuals, who are not teacher educators by training (e.g., mathematicians, adjuncts, graduate students), end up taking on a role of educating teachers when teaching mathematics content courses (Greenberg & Walsh, 2008). Moreover, these individuals do not have formal training in mathematics education

or preparing teachers to teach mathematics, nor do they have experience teaching mathematics to schoolchildren¹ (Bass, 2005; Hodgson, 2001; Sztajn, Ball, & McMahon, 2006). Furthermore, Masingila and colleagues (2012) documented that more than half of the mathematics department faculty who teach content courses feel unprepared and report lack of training, resources, and support at their institutions particularly related to the development of ePTs' knowledge about teaching mathematics to children.

Knowledge about children and their mathematical learning and thinking is complex and very difficult to address with ePTs (Ball, Thames, & Phelps, 2008; Carpenter, Fennema, Franke, Levi, & Empson, 1999; Hill, Ball, & Schilling, 2008). In this article, I particularly focus on this knowledge domain and aim to offer explicit practitioner examples documented in the literature about various learning opportunities that may be implemented with ePTs during mathematics content courses to help develop their knowledge about mathematical thinking and learning of schoolchildren.

Framework

Knowledge about mathematical thinking and learning of children is one of the critical domains of teachers' knowledge known as pedagogical content knowledge (PCK): a special type of teachers' knowledge that "represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interest and abilities of learners, and presented for instruction" (Shulman 1987, p. 8). Although PCK encompasses different knowledge domains (e.g., knowledge of instructional strategies, knowledge of curriculum, knowledge of students), experts argue that strengthening ePTs' knowledge of students and their mathematical thinking and learning is vital for the development of their professional identities and readiness to teach (An, Kulm, & Wu, 2004; Blömeke, Buchholtz, Suhl, & Kaiser, 2014; Grossman, 1990; Marks, 1990).

Specifically, knowledge of students and their mathematical thinking and learning involves teachers' expertise related to students' conceptions and misconceptions of particular topics, as well as their knowledge about specific approaches and strategies that help to address those conceptions and misconceptions effectively (Ball, Thames, & Phelps, 2008; Carpenter, Fennema, Franke,

Levi, & Empson, 1999). Teachers must be able to anticipate what students are likely to think, do, and find confusing regarding specific mathematical concepts. Ball, Thames, and Phelps (2008) further articulate:

When assigning a task, teachers need to anticipate what students are likely to do with it and whether they will find it easy or hard. Teachers must also be able to hear and interpret students' emerging and incomplete thinking as expressed in the ways that pupils use language. Each of these tasks requires an interaction between specific mathematical understanding and familiarity with students and their mathematical thinking (p. 401).

However, most opportunities designed around ePTs learning about students have been primarily documented in the mathematics methods courses often taught by the education department faculty (Greenberg & Walsh, 2008; Lutzer, Rodi, Kirkman, & Maxwell, 2007). Researchers argue that a methods course, completed by ePTs the semester prior to student teaching, is too late in the program and is not enough to help ePTs develop the necessary expertise and skills for anticipating, identifying, and addressing students' mathematics-specific conceptions and misconceptions. In fact, since most teacher education programs now offer and require mathematics content courses, designed to enhance ePTs' mathematical knowledge by having them make sense of concepts and principles that underlie the mathematics they learned as children (and will teach to children), these content courses offer an ideal platform for initiating ePTs' early experiences with students' mathematical thinking and learning, particularly as paralleling examples of ePTs' own childhood experiences in learning mathematics (Ambrose, 2004; Bass, 2005; Wideen, Mayer-Smith, & Moon, 1998). To address these efforts, in the past two decades, various textbooks also have been developed, and widely used in content courses nationwide, to specifically include examples of students' work, problem-solving strategies, transcripts and videos of students doing mathematics (e.g., Aichele and Wolfe 2007; Bassarear and Moss 2015; Beckman 2012; Fierro 2012; Musser, Peterson, & Burger, 2013; Parker and Baldrige 2004; Sowder et al. 2010).

Therefore, the goal of this study was to review mathematics education literature to offer explicit practitioner examples and recommendations in regards to embed-

¹ Note: In this paper, we use "children," "students," and "schoolchildren" to refer to elementary (age 5-12) pupils.

ding various learning opportunities into mathematics (elementary education) content courses to develop ePTs' knowledge of students and their mathematical thinking and learning. My aim with this article is to offer a variety of examples from the literature to help outline different types of learning opportunities that could be implemented by a college instructor, who teaches mathematics (elementary) content courses and has little/no prior experience teaching or working with schoolchildren, to be able to select specific examples appropriate for his/her class, delve deeper into the articles cited under those examples, and develop an implementation plan for those examples addressing his/her course settings, curriculum, and ePT population needs. The citations of the articles are especially helpful for novice college instructors in this area, offering an opportunity to contact the authors of the chosen articles for guidance and suggestions with implementation.

Methods

Data Sources

This work is primarily based on review of mathematics education literature published in the last two decades (1998-2018). Most of the articles identified for this study are sourced from the National Council of Teachers of Mathematics (NCTM) journals. The articles were selected based on their explicit practitioner nature, including examples, implications, and evidence of specific learning opportunities designed to develop ePTs' knowledge of students and their mathematical thinking and learning.

Data Analysis

To target the selection of articles that discuss classroom opportunities in *mathematics content courses* for developing *prospective (elementary) teachers'* knowledge about *schoolchildren*, I used Boolean multi-database searches using different "and/or" combinations of specific keywords (e.g., mathematics "and" children learning "and" preservice teachers; also prospective teachers, student learning, mathematics courses). This search identified 514 potentially relevant journal articles. The majority (340) of these articles were published in the NCTM journals (e.g., *Mathematics Teacher Educator*, *Mathematics Teaching in the Middle School*, *Teaching Children Mathematics*) and about a third (174) were manuscripts published in books, monographs, and other mathematics education journals, which included practice-based recommendations (e.g., *Journal of Mathematics Teacher Education*; *School Science and Mathematics*; *International*

Journal of Educational Research). I read the abstracts of all resultant (514) manuscripts to identify specific ones for full review that were relevant to the topic (classroom opportunities in mathematics content courses for developing ePTs' knowledge of students and their mathematical thinking and learning). The final number of the scholarly products reviewed as part of the literature for this study was forty-nine (49) total.

Due to the small sample size and nature of this study, I conducted a qualitative thematic literature review, identifying common trends prevailing in the literature and offering specific classroom-based examples of learning opportunities (around those themes) specifically designed to help develop ePTs' knowledge of students and their mathematical thinking and learning. For example, in the initial literature review, I identified two major themes that addressed developing ePTs' knowledge of schoolchildren and their mathematical thinking and learning via direct and indirect interactions with them. Furthermore, within the "direct interactions" literature, I further identified several clusters of research articles situated around specific focuses: interviews, quick written assessments, family nights and afterschool projects, or math pen-pal activities. Similarly, within the "direct interactions" literature, I identified several clusters of articles addressing specific focuses: transcripts and videos, correct and incorrect written students' work, or hypothetical situations that required ePTs responses to children's mathematics. Below, I describe specific examples from these literature-based themes and provide verbatim quotes and descriptions directly from the articles as helpful (representative) and practical exemplars.

Results

Developing ePTs' Knowledge of Students and their Mathematical Thinking and Learning:

Direct Interactions

One of the largest themes in this category was interviews with children, in which many manuscripts included direct evidence for the interviews strengthening ePTs' mathematical and pedagogical knowledge related to children's problem solving and thinking strategies (see Fernandes, 2012; Friel, 1998; Gee, 2006; Jenkins, 2010; Lannin & Chval, 2013; McDonough, Clark & Clark, 2002; Spangler & Hallman-Thrasher, 2014). These studies also strongly suggested that, to be effective, the interviews must include a well-defined goal, purpose, and structure for ePTs to accurately follow, assess, and reflect upon children's thinking.

For example, Friel (1998) engaged ePTs in a four-task interview to help them determine children's understanding of the concept of average (arithmetical mean). These interviews involved low- and high-level tasks. The ePTs were given a specific protocol to follow: interview a small group of children (individually) about their thinking as they try to solve these tasks; listen to and keep records of their (different) strategies. Similarly, Spangler and Hallman-Thrasher (2014) reported ePTs working directly with children by posing mathematical problems to them while observing and gaining insights into how they think about numbers and operations. Both studies reported ePTs mathematical and pedagogical growth from these experiences in their ability to work with students, interview them, and pose specific questions that help to gather insightful information about students' mathematical thinking, reasoning, and problem-solving strategies.

Additionally, Gee (2006) described the "math-mates" structured interviews embedded into her mathematics courses in which she focused ePTs' attention on children explaining their thinking while ePTs asked open-ended follow-up questions to probe students' thinking. The examples of open-ended questions ePTs asked children included the following: *How did you get that answer? Can you show me how you solved that problem?* Similarly, Whitin and Whitin (2003) found that engaging ePTs in asking children open-ended questions, including divergent discussion prompts, were most effective for ePTs in learning about how to support children in articulating mathematics and in their thinking, answers, and solution paths.

Johnson, Campet, Gaber and Zuidema (2012) reported that virtual-manipulatives interviews also engage ePTs in helping students articulate their thinking. The authors reported that when conducting clinical interviews with students, the ePTs were specifically instructed to ask "manipulatives-based" interview questions, such as: *Can you use the pieces on the screen to show me what you are doing/thinking for this problem? Why are you [shading that in, or moving that part, or selecting that portion] of the object? How does that help you solve the problem?* Johnson and colleagues (2012) shared that these clinical interviews helped ePTs learn how to carefully select meaningful tasks, corresponding virtual manipulatives, and observe and probe "in action" students' problem-solving strategies for using virtual tools/manipulatives.

Additionally, a number of studies included descriptions of ePTs' direct interactions with children involving non-interview opportunities, including the following: administering written prompts, questions, and assessments to children, organizing after-school projects and

math-nights, and engaging in pen-pal activities (see Lampe & Uselman, 2008; Sjöberg, Slavit, & Coon, 2004; Shockey & Snyder, 2007; Stephens & Lamers, 2006). For example, quick assessments, prompts, and discussion questions administered to students enabled ePTs to gain insights into their thinking and gauge students' prior and existing mathematical knowledge. Administered assessments included a variety of tasks, including take-home or in-class prompts, tasks and problems, projects, quizzes/tests, and exit slips.

Most of these studies also demonstrated strong evidence of the assessment-development activity itself helping ePTs to better understand how the nature of the question/task influences the information gained from students' responses (e.g., more information might be gained from an open-ended rather than a multiple-choice question). For example, Stephens and Lamers (2006) identified specific prompts that helped to guide the ePTs' assessment-development skills, including the following: *Does the task specifically assess your chosen content area and the conceptual understanding of the content, rather than merely procedural skill? Does the task elicit different representations or strategies? Will students' responses provide you with valuable feedback about their thinking?* (Stephens & Lamers, 2006, p. 119).

Sjöberg, Slavit, and Coon (2004) also described that "it was helpful to [ePTs] to convey clear expectations, demonstrate effective writing in mathematics, and provide specific prompts" to help elicit children's thinking and reflect upon their responses. The authors shared the following prompts that they used with their ePTs: *What? [What have you learned] So what? [What difference did it make] Now what? [What can you do with this information]* The authors argued that assessment-development skills provided ePTs with a critical knowledge of being able to gain insights on students' thinking and helped when ePTs were making specific observations about students' learning and progress (Sjöberg, Slavit, & Coon, 2004, p. 490).

A number of studies also elaborated on organizing family math nights and afterschool projects as added in-school experiences for ePTs to directly engage with students and observe their learning (e.g., Bofferding, Kastberg, & Hoffman, 2016; Freiberg, 2004; Lachance, Benton, & Klein, 2007; Lachance, 2007; Shockey & Snyder, 2007). These experiences often involved different groups: students and their families, teachers, administrators, school staff, and/or college mathematics instructors. The primary goal for these groups was to have an opportunity "to learn about mathematics together in an informal and supportive setting" and gain a better ap-

preciation for “how mathematics is integrated into a variety of contexts and subject areas while having fun” (Lachance, 2007, p. 407). However, most popular math nights activities (discussed in the literature) involved stations, where a pair of ePTs would have their own activity table structured in the form of a contest or game for students to play and compete against themselves, their peers, parents, or even their ePT hosts (e.g., Freiberg, 2004; Lachance, Benton, & Klein, 2007; Lachance, 2007).

Other forms of afterschool projects also report fruitful learning experiences for ePTs and schoolchildren (and college instructors), who typically engage in mathematics activities with the purpose of helping schoolchildren to complete a mathematically rich task/activity (individually or in small groups). For example, Shockey and Snyder (2007) described a “tessellation t-shirts activity” completed with ePTs and K-4 students to offer additional opportunities for ePTs to explore and observe how young children use hands-on experiences to explore geometric patterns. The authors described, “After students have repeatedly traced their tessellating design on a large piece of newsprint, they color their design with fabric crayons. Such designs can be ironed onto polyester or cotton-blend T-shirts... The college instructors and ePTs operate the ironing stations.” (Shockey & Snyder, 2007, p. 86).

Finally, math pen-pals activities seem to be gaining wide popularity as well (Appova & Taylor, 2017; Crespo, 2000; Lampe & Uselman, 2008; Phillips & Crespo, 1996). A pen-pal activity typically involves ePTs and students exchanging “math” letters related to “custom-made” (grade-appropriate and aligned with standards and mathematics curriculum) mathematically-rich tasks and problems, for which students receive personalized feedback from ePTs on their solutions. For example, Lampe and Uselman (2008) reported that, through a pen-pal activity, ePTs made meaningful connections with students in a “quasi-teaching capacity” which then allowed the ePTs put their knowledge into practice. The school students put greater-than-usual effort as well in developing solutions and “trying more avenues and methods to reach a coherent solution rather than simply guessing or giving up” (Lampe & Uselman, 2008, p. 200). Similarly, Crespo (2003) argued that a pen-pal activity that she implemented in her course, encouraged the ePTs and their young counterparts “to make explicit their mathematical thinking, deepen their own understanding of the subject, and become more comfortable doing and talking about mathematics” (p. 34). The particular appeal of various pen-pal activities is in the convenience as the pen-pal

system is somewhat of a hybrid model: it offers ePTs direct access to students’ mathematics via indirect interactions with them (Appova & Taylor, 2017).

Developing ePTs’ Knowledge of Students and their Mathematical Thinking and Learning: Indirect Interactions

If direct access to schoolchildren is not feasible to embed into the course, other (indirect) opportunities are still available for developing ePTs’ knowledge about children’ mathematical learning, particularly using videos and authentic artifacts collected from students (e.g., work samples, excerpts, written solutions). In fact, the largest theme in this category involved articles describing different databases and resources readily available with videos, transcripts, and case studies of schoolchildren (e.g., Cognitively Guided Instruction: Integrating Research on Teaching and Learning Mathematics [CGI]; Integrating Mathematics and Pedagogy to Illustrate Children’s Reasoning [IMAP]; Annenberg Video Series; Teaching Channel; Show-Me Center). Many of these projects actually offer guidelines and suggestions on how to implement the resources with ePTs. For example, the CGI project includes specific guidelines on how to use their project videos and materials with ePTs, how to guide ePTs to learn about specific problem types for whole number operations, and how to interpret different strategies and levels of thinking that elementary students utilize to solve those problems (Carpenter, Fennema, & Franke, 1996).

In addition to analyses of students’ work, several projects offer video resources that can help college instructors examine and assess the knowledge of ePTs. For example, the IMAP project offers an assessment instrument to help measure teachers’ beliefs about teaching and learning, as well as the nature of mathematics, depth of their mathematical knowledge, and their knowledge about schoolchildren. Specifically, this beliefs assessment “consists of video clips of students with an accompanying questionnaire” that can be used to study teacher change and to develop course materials for mathematics (elementary) in-service and prospective teachers (see project page: <http://www.sci.sdsu.edu/CRMSE/IMAP/overview.html>).

Several studies also reported the use of children’s videos to help promote ePTs’ understanding of specific issues related to equity, social justice, and culturally responsive pedagogy. For instance, Jilk (2016) used video case analyses to encourage ePTs to look for and pay close attention to the resources (what students have) and the potentials of students (what they know and are able to

do). It has been documented that when ePTs learn to notice and focus on students' strengths, capabilities, and conceptions, they begin to perceive and approach students as competent learners (Cohen, 1994). Furthermore, it is through the observations of "students' strengths" that the teachers are able to provide better support to students and develop their positive mathematical identities (de Abreu & Cline, 2007; Jilk, 2014; Martin, 2000, Nasir, 2002), as well as broaden teachers' pedagogical repertoire of classroom strategies to include a rich set of mathematical problems and skills (Boaler & Greeno, 2000). Experts suggest that these aspects of teaching are especially critical and necessary for creating a robust instructional program for all schoolchildren, particularly "young people who have traditionally been marginalized by school mathematics" (Jilk, 2016, p. 188).

Besides the videos, several studies also described ePTs examining students' thinking and learning by analyzing written solutions and work samples collected from students. For example, Herbel-Eisenmann and Phillips (2005) asked ePTs to solve and sort 16 different algebra problems based on the characteristics noticed in each problem. Next, after sharing and discussing their solutions, ePTs were presented with students' work from the same sixteen problems and were asked to articulate what they noticed and learned about students' mathematical understanding in each problem. By examining students' work, the ePTs became "more aware of the mathematics embedded in the problem and the diversity of students' algebraic reasoning" (Herbel-Eisenmann and Phillips 2005, p. 63). Likewise, Cianca (2013) engaged ePTs in analyzing students' drawings of geometric figures (e.g., pictorial, schematic, isometric) and reported that this activity not only increased ePTs' knowledge of geometry, but also helped them to develop "noticing skills" for analyzing students' work and (mathematically) critiquing their geometric diagrams.

A number of studies also reported ePTs analyzing students' work for mathematical errors, which particularly helped ePTs to appreciate and understand students' misconceptions and reason about their (incorrect) work, as well as being able to mathematically critique children's reasoning (see Borasi, 1994). For example, Lim (2014) selected children's work that included specific types of errors and asked ePTs to analyze and discuss children's misconceptions. These experiences offered the ePTs opportunities "to tackle certain problems head-on, discuss the mathematics underlying the errors, learn from these mistakes, and deepen their own mathematical understanding" (Lim, 2014, p. 111). The author also suggested selecting specific children's work that in-

cluded "error-eliciting" characteristics. Lim (2014) described that the work must bring forth "common" mistakes that children often make pertaining to a particular mathematical concept, including the work that can help ePTs (and college instructors) to note and extract specific misconceptions, misapplication of a procedure or a formula, or overgeneralization of a concept.

Finally, Lannin and Chval (2013) argue that, if none of the aforementioned direct and indirect interactions with children are feasible in the course, university instructors could additionally offer ePTs opportunities to "keep in mind" children as the ePTs themselves are learning mathematics during content courses. For example, the authors suggest framing a mathematics question, problem, or task within the context of children's thinking. The authors shared a quick example as an excerpt from their class:

Your third graders are playing "double-compare" that involves multiplication, where each student chooses two cards, multiplies the numbers on the card, and compares the answer to the results of others. Michelle notices that when someone has the "same cards" (e.g., 4 and 4), the result is always one more than a student who has cards that are one more and one less than the "same cards" (e.g., 3 and 5). Is this true for all whole numbers? Justify your response. Also, if this situation occurred in your classroom, how would you respond? (Lannin & Chval 2013, p. 510)

Lannin and Chval (2013) particularly argued that framing a task within the context of children's thinking helps to better motivate and engage ePTs in the task and requires them to consider appropriate (teaching and learning) strategies, including a specific criteria for identifying valid mathematical reasoning, representations, and argumentations.

Conclusions and Discussions

One responsibility of university instructors who teach courses to ePTs is to develop the necessary knowledge for them to become highly qualified professionals and effective mathematics educators (Hallett, Nunes, & Bryant, 2010; Hiebert & Lefevre, 2013; Thanheiser et al., 2014). Experts argue that university faculty who teach mathematics to ePTs not only need to know mathematics content, but they also need to know how to engage ePTs in the development of mathematics-specific pedagogical

skills, as well as mathematical knowledge, to help ePTs become better prepared to teach mathematics to children (Ambrose, 2004; Bass, 2005; Masingila et al., 2012; Wideen, Mayer-Smith, & Moon, 1998).

Substantial evidence indicates that learning opportunities specifically designed around developing knowledge about children and their mathematical thinking and learning directly support ePTs' abilities to reflect on classroom instruction and improve student learning (e.g., Baumert et al., 2010; Burton, Dane, & Giessen, 2008; Capraro, Capraro, Parker, Kulm, & Raulerson, 2005; Carpenter, Fennema, Peterson, & Carey, 1988; Hill, Ball, & Shilling, 2008; Tirosh, 2000; Tirosh, Tsamir, Levenson, & Tabach, 2011; Vale, 2010).

In this paper, I provided a literature-based inventory (e.g., lesson ideas, activities, excerpts, questions/prompts, resources) related to various learning opportunities embedded into mathematics content courses, designed specifically for ePTs, for the purpose of enhancing and developing their knowledge of/about children and their mathematical thinking and development. This literature-based inventory strongly demonstrates that a wide range of learning opportunities are being implemented in the field (by colleagues) and are readily available to college instructors to help address this particular knowledge domain in the content courses. Furthermore, this literature-based inventory demonstrates that many of these learning opportunities do not require college instructors' prior experiences to include K-12 teaching or working with children or for their content courses to include direct access to schoolchildren.

In fact, extant literature suggests that indirect interactions with students are comparable to direct interactions, particularly in developing the necessary *noticing skills* in ePTs and engaging them in critical mathematical and pedagogical analyses of children's videos and mathematics work samples (e.g., Amador, 2017; Carpenter et al., 1999; Jilk, 2016; McDuffie et al., 2014). For example, Amador (2017) utilized animated videos of children to help ePTs develop *noticing skills* related to the content of children's understanding of fractions. She engaged ePTs in discussions about *who* (students and teacher) and *what* (concepts and pedagogical moves) they notice while watching those videos. Similarly, McDuffie and colleagues (2014) studied video analysis activities and found significant improvements in ePTs' *noticing* of students' multiple knowledge bases and levels of thinking via four critical lenses: teaching, learning, mathematical tasks, and participation. These studies strongly suggest that both direct and indirect interactions with children offer fruitful support structures for developing ePTs' pro-

fessional noticing skills and critical lenses for making sense of children's work by "attending to children's strategies, interpreting children's understandings, and deciding how to respond on the basis of observed children's understandings" (Jacobs, Lamb, & Philipp, 2010, p. 172; also see Philipp, 2008; Thomas, Fisher, Jong, Schack, Krauseand, & Kasten, 2015; Sherin & van Es, 2003).

Ideally, if a content course allows for *direct* access to schoolchildren, research shows that these types of interactions not only help ePTs to develop awareness of the specific strategies and thinking that children may apply when solving mathematics problems, but these experiences also facilitate ePTs' ability to adopt *interpretative lenses* (rather than *evaluative* lenses) when working with children and examining their solutions (Crespo, 2000; Mason, 2002). Furthermore, studies show that direct interactions with students help ePTs gain instrumental experiences in working with "real" students, learning how to pose questions, anticipate answers, respond to children's thinking, make mathematical connections between students' thinking and mathematical topics, and become more experienced in addressing students' conceptions and misconceptions in "real time" and on the spot. Lannin and Chval (2013) call these opportunities "powerful" because they provide ePTs with firsthand insights into students' learning and embed experiences for ePTs to try out various instructional strategies directly with students. Most importantly, these opportunities allow ePTs to recognize how difficult it is to gain insights into students' thinking, particularly the challenge of selecting a meaningful task or a question and being able to draw accurate conclusions about children's knowledge (Lannin & Chval, 2013).

Ultimately, by offering this review of extant literature, I argue that university faculty who teach mathematics content courses to ePTs, including the instructors who do not have experiences teaching or working with K-12 students, have a plethora of readily available resources and classroom-based examples (from the field) to help them embed course activities that focus ePTs' attention and reflection (directly or indirectly) on students' mathematical learning and thinking. Research supports evidence that these course activities improve ePTs' readiness to teach and their preparedness to work with schoolchildren, particularly appreciating students' mathematical successes and difficulties. Embedding these course activities also helps to strengthen ePTs' abilities to "anticipate students' responses, address and redirect partially correct and incorrect responses, and match follow-up questions and suggestions to their students' thinking" (Spangler & Hallman-Thrasher, 2014, p. 63).

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