JOURNAL OF

# MATHEMATICS EDUCATION

AT TEACHERS COLLEGE

*A Century of Leadership in Mathematics and Its Teaching*

## Mathematics Teacher Education

# TABLE OF CONTENTS

JOURNAL OF **MATHEMATICS EDUCATION** AT TEACHERS COLLEGE | SPRING 2015 | VOLUME 6, ISSUE 1



# Patterns in the Pythagorean Configuration and Some Extensions: The Power of Interactive Geometry Software

José Contreras
*Ball State University*

**ABSTRACT** In this paper I describe classroom experiences with pre-service secondary mathematics teachers (PSMTs) investigating and extending patterns embedded in the Pythagorean configuration. This geometric figure is a fruitful source of mathematical tasks to help students, including PSMTs, further develop habits of mind such as visualization, experimenting, looking for and noticing patterns, conjecturing, inventing, constructing mathematical arguments, and posing problems. By carrying out these tasks, the PSMTs were also engaged in a plethora of mathematical practices recommended by the Common Core State Standards Initiative (CCSSI). The use of the Geometer's Sketchpad facilitated the accomplishment of most of these activities. As students of mathematics, these PSMTs were engaged in activities to reinforce some of their mathematical habits of mind and experienced learning new mathematical ideas and processes through practices that exemplify typical mathematical thinking. As future teachers, they were engaged in tasks so they can, in turn, design or adapt instructional tasks to develop further their own students' habits of minds and engage them in learning mathematics through the mathematical practices advocated by the CCSSI.

**KEYWORDS** *conjectures, experimenting, extending, generalizing, habits of mind, Interactive Geometry Software, problem posing, Pythagorean configuration, patterns, visualization*

As a mathematics teacher, one of the most exciting rewards is to help my students (in-service and pre-service middle school and secondary mathematics teachers) experience the delight of discovering, formulating, and generalizing patterns, conjectures and theorems. These processes lie at the very heart of doing mathematics (Cuoco, Goldenberg, & Mark, 1996; Brown & Walter, 1990; Halmos, 1980; Polya, 1973).

Cuoco, Goldenberg, and Mark (1996) argue that school mathematics should provide students opportunities to develop habits of mind including visualizing, searching for patterns, experimenting, conjecturing, inventing, and constructing mathematical arguments. In order for teachers to design tasks that have the potential to foster these habits of mind in their students, they should themselves possess and appreciate such habits. Thus, it is critical that our pre-service mathematics teachers have experiences performing activities to develop or reinforce them.

Other scholars (e.g., Brown & Walter, 1990; Halmos, 1980; Polya, 1973) argue that students should have ample experiences in posing problems related to a given problem. Generalizing is a typical and useful problem-posing strategy that can be applied to a wide range of mathematical situations. Again, if we want all students to be able to create mathematical problems, then their teachers themselves should be problem posers. Thus, pre-service teachers should be given tasks to develop or reinforce their knowledge, abilities, and dispositions to generate problems.

More recently, reform efforts led by the National Governors Association Center for Best Practices (NGA) and the Council of Chief State School Officers (CCSSO) resulted in the development of the Common Core State Standards Initiative (CCSSI). This initiative calls for students to learn mathematics through a variety of practices including looking for and making use of structure, using appropriate tools strategically, constructing viable arguments and critiquing the reasoning of others, and looking for and expressing regularity in repeated reasoning. If we want student to learn mathematics through these practices, then their teachers should be confident in their knowledge and abilities to do so. Thus, it is crucial that pre-service teachers experience learning mathematics through said practices or processes.

A tool that can serve to facilitate some of these important mathematical habits, processes, and practices is Interactive Geometry Software (IGS) such as The Geometer's Sketchpad ((GSP), Jackiw, 2001), Cabri Geometry (Laborde & Bellemain, 2005), and GeoGebra ((GG), Hohenwarter, 2002), to name just a few.

Some scholars (e.g., Contreras, 2004, 2009; Laborde & Laborde, 1995) describe some of the learners' behaviors promoted by teaching and learning mathematics using IGS. First, IGS allows learners to construct a geometric configuration faster and more precisely than using physical tools such as straightedge and compass. Second, IGS aids learners to visually notice a pattern and to wonder whether the pattern seems plausible. Third, learners are able to use one or more features of IGS (e.g., measurement, construction, etc.) to verify the pattern and then formulate an initial conjecture. Fourth, IGS can allow students to drag flexible points of the configuration to test whether the conjecture is valid in other cases.

In this paper I describe some classroom experiences in which groups of prospective secondary teachers, henceforth referred to as students, have investigated patterns embedded in the Pythagorean configuration. The students, juniors and seniors, are mathematics majors enrolled in a geometry course for teachers.

The Pythagorean configuration affords opportunities for students in general, and pre-service teachers in particular, to develop some habits of mind, abilities to pose problems, and other mathematical processes or practices, all of which are mentioned previously. The constructions were performed with GSP or GG but other types of IGS can be used. In this article all figures are constructed with GSP.

## The Pythagorean Configuration

I often ask my students to explore geometric patterns embedded in the Pythagorean configuration (Figure 1). As its name suggests, the Pythagorean configuration consists of a right triangle, the squares constructed on its sides and, in addition, other objects determined by elements displayed in Figure 1.

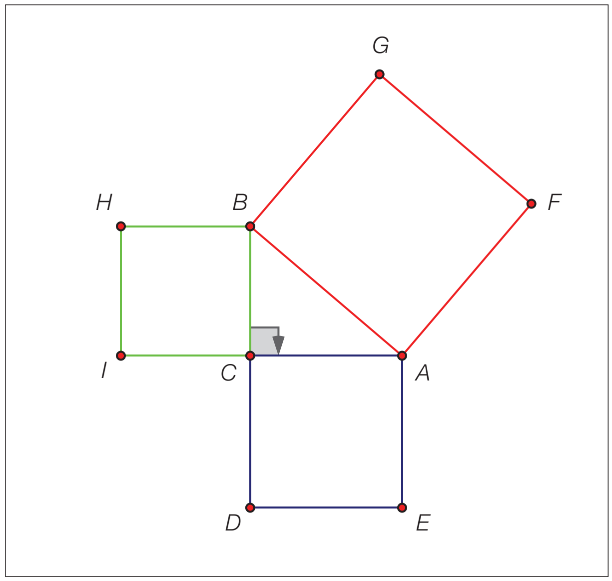


*Figure 1.* The Pythagorean configuration.

Students are asked to construct other geometric objects determined by the given elements of Figure 1. Sometimes this task is one of the first investigations that a particular class performs. In other cases, students have already had experiences exploring other configurations (e.g., Napoleon's configuration). Some of the most interesting constructions that students have proposed are displayed in Figure 2.

## The Pythagorean Configuration, Triangles, Areas, and Generalizations

After students have constructed the configuration displayed in Figure 2a, I asked them to examine it and describe, in writing, any patterns that they may notice. Some of the conjectures that students have initially formulated are as follows:

(a) Triangles CDI and CAB are congruent and, hence, have the same area;
(b) Triangles AEF and BHG are congruent and, hence, have the same area.

Before students jump to use GSP to verify their conjectures, I often ask them to think whether the conjectures are plausible. My goal is that students further develop their habit of analyzing mathematically the verisimilitude of a conjecture before using technological tools to refute it. Some students immediately realize that if triangles AEF and BHG were congruent it would imply that ΔABC is an isosceles (right) triangle. In contrast, other students correctly justify that triangles CDI and CAB are always congruent when ΔABC is a right triangle.

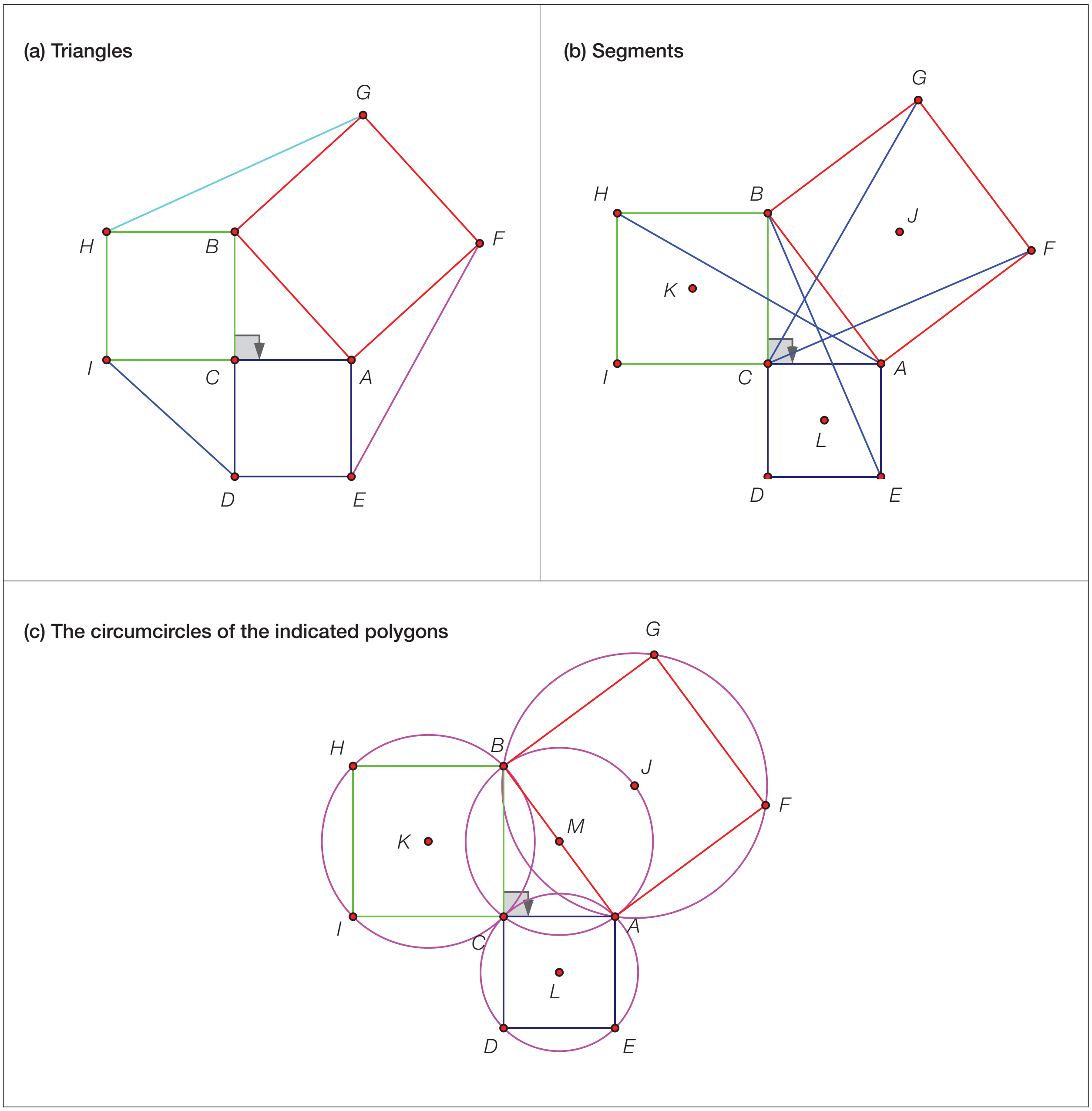


*Figure 2.* Some objects determined by elements of the Pythagorean configuration.

As students refute the conjecture that triangles AEF and BHG are congruent, some visualize that their areas may be equal (Figure 3). To test their refined conjecture, some students use IGS to measure the areas of the two triangles and others measure also the areas of the two right triangles. Notably, the areas of the four triangles AEF, BHG, CDI, and ABC seem to be equal (Figure 4).

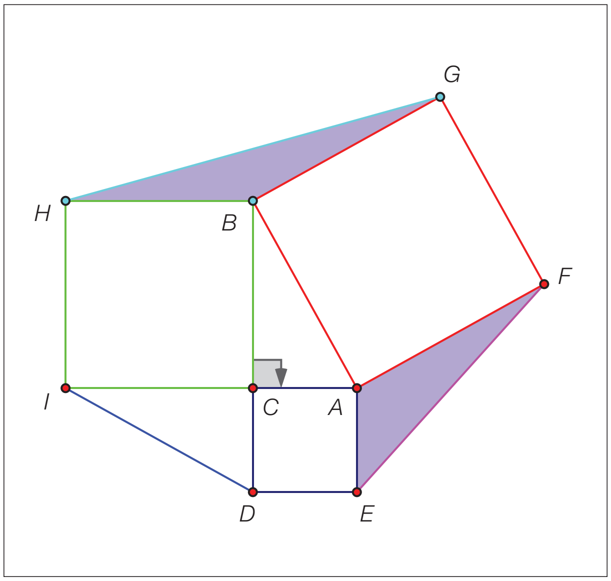


*Figure 3.* The areas of triangles AEF and BHG seem to be equal.

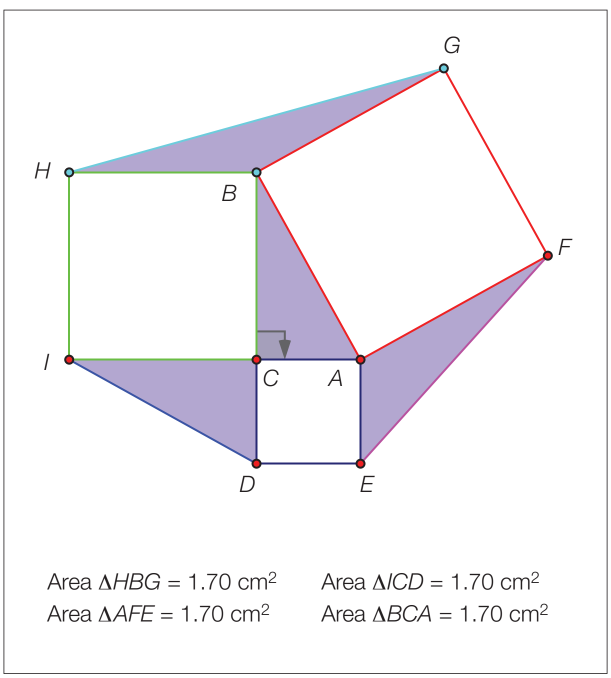


*Figure 4.* The areas of the four triangles seem to be equal.

After students test the conjecture for a wide range of triangles by effortlessly dragging flexible points of the right triangle, some are motivated to develop a proof. Of course, a few students would argue that the measurements and the multiple (empirical) examples constitute a proof. A proof that the four triangles AEF, BHG, CDI, and ABC are equivalent (i.e., have the same area) is displayed in appendix A.

To enhance my students' problem posing abilities, I regularly ask them to try to generalize or extend mathematical results. A possible generalization of the previous result involves considering arbitrary triangles instead of right triangles (Figure 5). To students' surprise, GSP shows that the areas of the four triangles ABC, CDI, AEF, and BGH are the same in the general case. Equally surprising was the fact that the proof extends smoothly for the general case. At this point students realize that the original argument did not depend on triangle ABC being a right triangle, and thus is valid for non-right triangles.

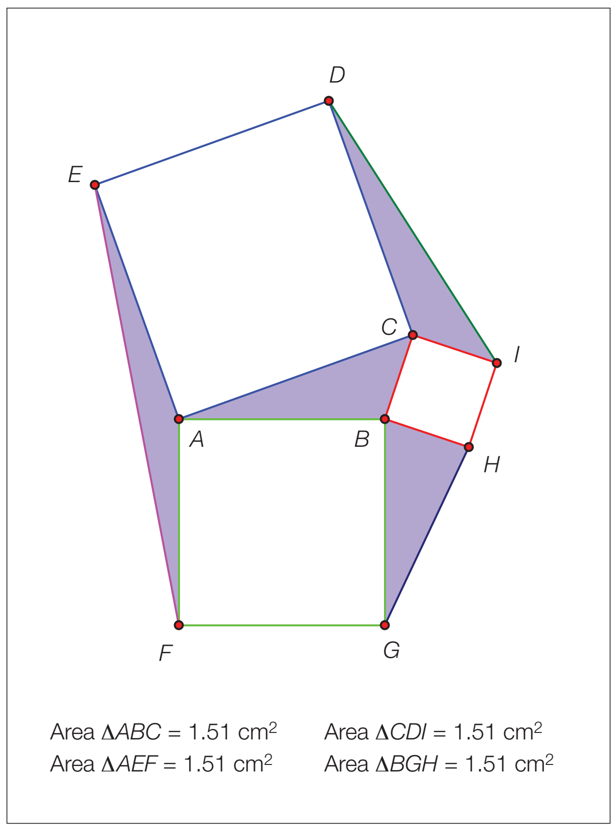


*Figure 5.* The areas of the four triangles are equal for an arbitrary triangle ABC.

### The Pythagorean Configuration, Segments, and a Generalization

As students examine the configuration displayed in Figure 2b, some hypothesize that the two pairs of segments ($\overline{CF}$ and $\overline{BE}$, $\overline{AH}$ and $\overline{CG}$) are congruent while others conjecture that they are also perpendicular. Measuring the relevant objects and dragging flexible points confirms their conjectures (Figure 6).

At this point the class is asked to develop a proof. A proof that segments $\overline{CF}$ and $\overline{BE}$ are congruent is based on the fact that ΔCAF ≅ ΔEAB by the SAS congruence criterion ($\overline{AF} \cong \overline{AB}$, ∠FAC ≅ ∠BAE, and $\overline{AC} \cong \overline{AE}$, Figure 7). More challenging is to prove that the segments are perpendicular. If nobody offers a suggestion, I provide a strategic hint to students: Look for two similar triangles, one of which you know has a right angle. After some reflection, some students propose triangles AFP and MBP (or AEO and MCO) (Figure 7). The triangles of these two pairs are similar by the angle angle similarity principle. For example, ΔAFP ~ ΔMBP because ∠APF ≅ ∠MPB (vertical angles are congruent) and ∠AFP ≅ ∠MBP (corresponding angles of congruent triangles are congruent). Finally, *m*(∠BMP) = *m*(∠FAP) = 90°. A similar argument shows that segments $\overline{AH}$ and $\overline{CG}$ are congruent and perpendicular.

As is often the case in an inquiry-based classroom, students then investigate whether the congruence and perpendicularity of the pairs of segments ($\overline{CF}$ and $\overline{BE}$, $\overline{AH}$ and $\overline{CG}$) hold for general triangles. Some students notice that the proof extends to general triangles while others use GSP to determine a possible generalization. The use of GSP leads students to surprisingly discover that there are actually three pairs of congruent and perpendicular segments in the generalized Pythagorean configuration (Figure 8).

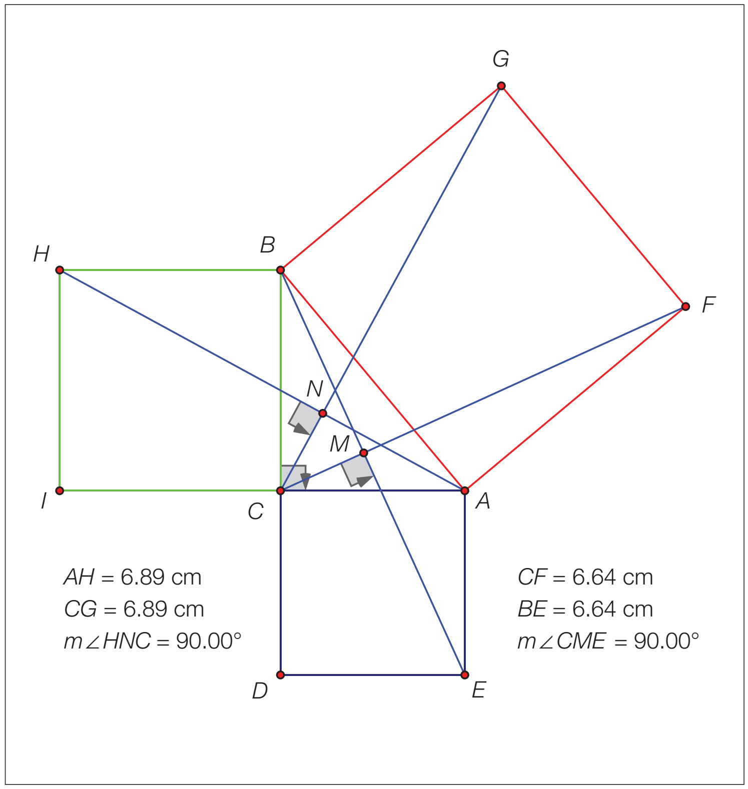


*Figure 6.* Some congruent and perpendicular segments.

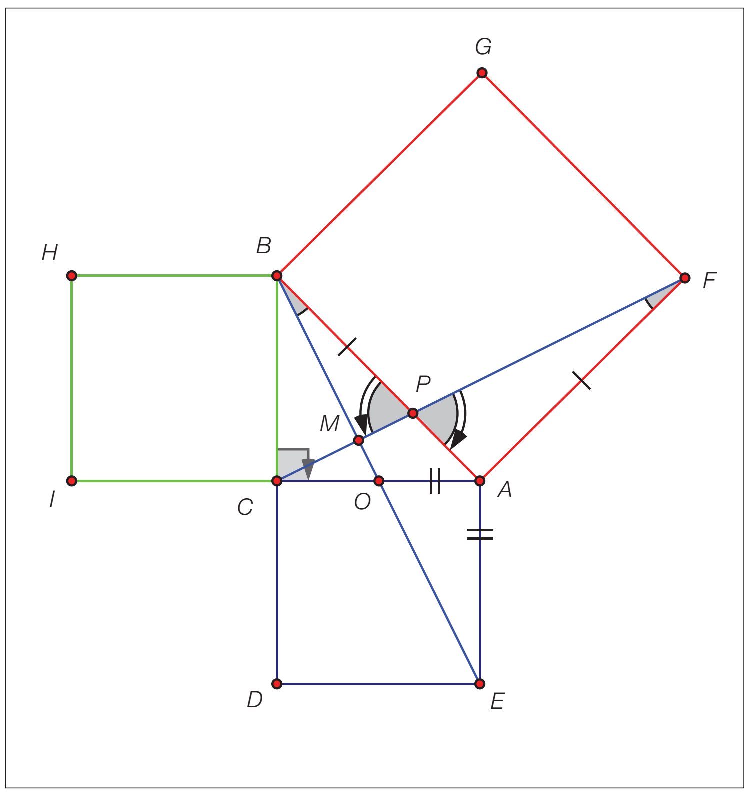


*Figure 7.* Diagram to prove that segments $\overline{CF}$ and $\overline{BE}$ are congruent and perpendicular.

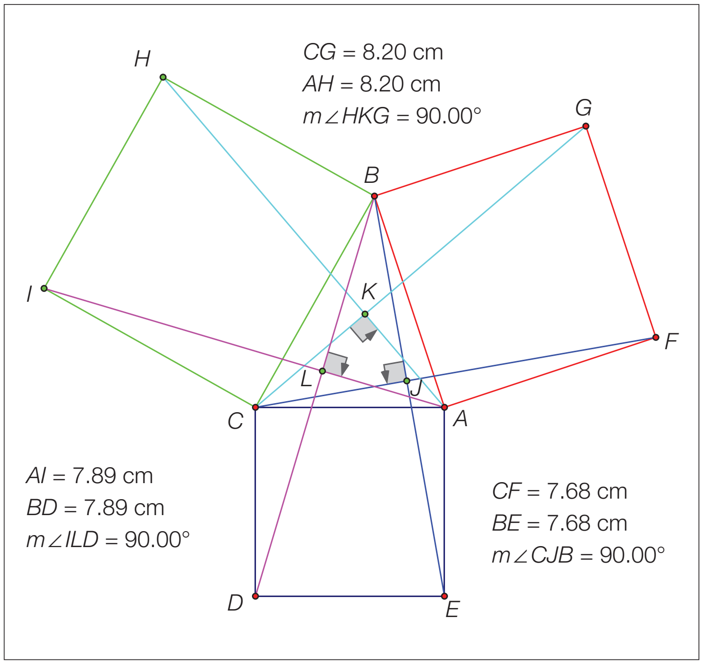


*Figure 8.* Three pairs of congruent and perpendicular segments in an arbitrary (acute) triangle.

### The Pythagorean Configuration, the Circumcircles of the Polygons, and an Extension

Another under-explored yet worthwhile pattern embedded in the Pythagorean configuration involves the circumcircles of the four polygons (Figure 2c). After students construct the circumcircles of the squares and the right triangle, they examine the configuration and write any patterns that they notice. Some of the patterns that students have noticed in Figure 2c are the following:

(a) The center of the circumcircle of the right triangle is on the hypotenuse;
(b) The circumcircle of the right triangle goes through point J, the center of the square with side $\overline{AB}$ ;
(c) Two of the circumcircles are tangent at C;
(d) Points K, C, and L are collinear;
(e) The circumcircles of any two squares and the circumcircle of the right triangle are concurrent.

Our first task is to discuss which of these patterns are significant. After some deliberation, students generally agree that pattern (e) is trivial because, by definition, the circumcircle of a polygon goes through its vertices. For example, point B is contained in the circumcircle of square BCIH, the circumcircle of square ABGF, and the circumcircle of the triangle. After this discussion, the class proceeds to prove their conjectures. They are either assigned a conjecture or choose one. Appendix B displays a proof for conjecture (b).

To prove that circles with centers K and L are tangents, some students construct the perpendicular to segment $\overline{KL}$ that goes through C (points K, C and L are collinear, Figure 9). Because a perpendicular to a radius at the outer endpoint is tangent to the circle, we conclude that such a perpendicular is the common tangent to the two circles. Thus, the two circles are tangent.

As the class further examines figure 9, some students notice that the common tangent of the two circles goes through point J, the center of the square associated with the hypotenuse of the triangle. As the class tries to develop a proof some students realized that $\overleftrightarrow{CJ}$ is in fact the angle bisector of the right angle [$m(\angle BCJ) = m(\angle KCB) = m(\angle LCA) = m(\angle ACJ) = 45°$]. A proof of this is displayed in Appendix C.

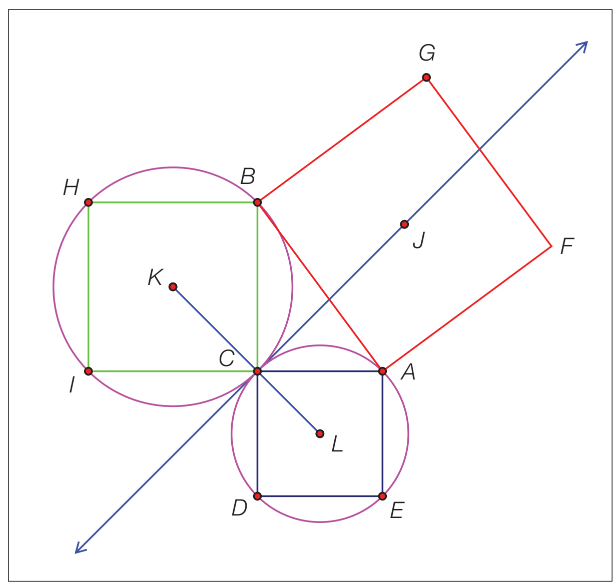


*Figure 9.* The two circles are tangent.

Our next task is to try to generalize or extend some of the results associated with configuration 2c. To accomplish this goal, students construct the configuration displayed in Figure 10.

As the class examines the configuration displayed in Figure 10, the students often do not notice any apparent

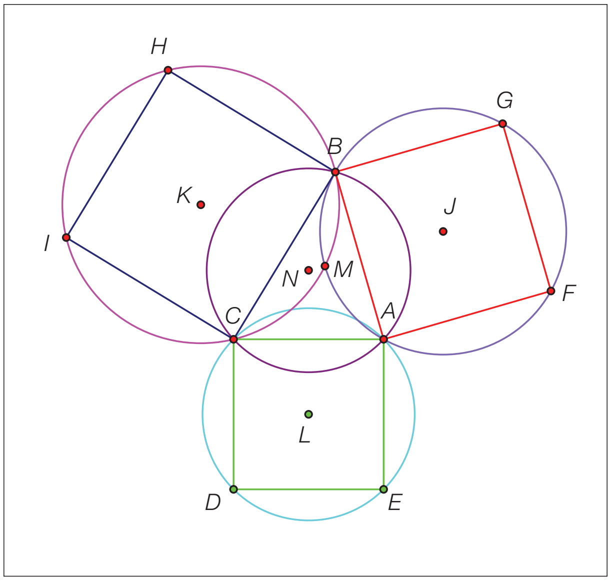


*Figure 10.* The circumcircles of the four polygons.

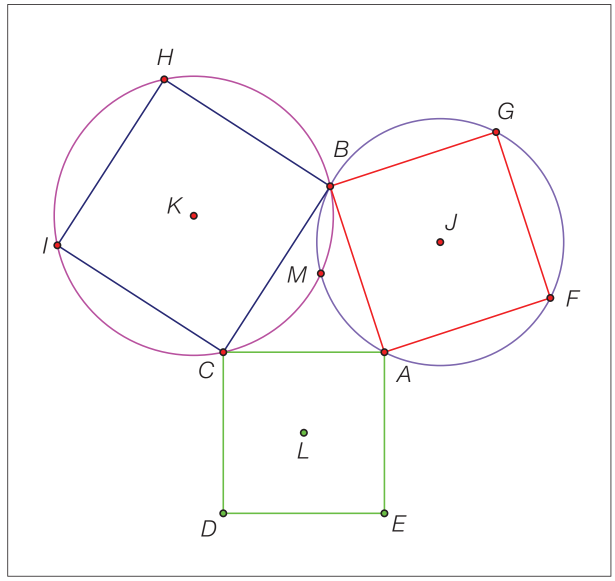


*Figure 11.* Is there a circle related to square ACDE and containing point M?

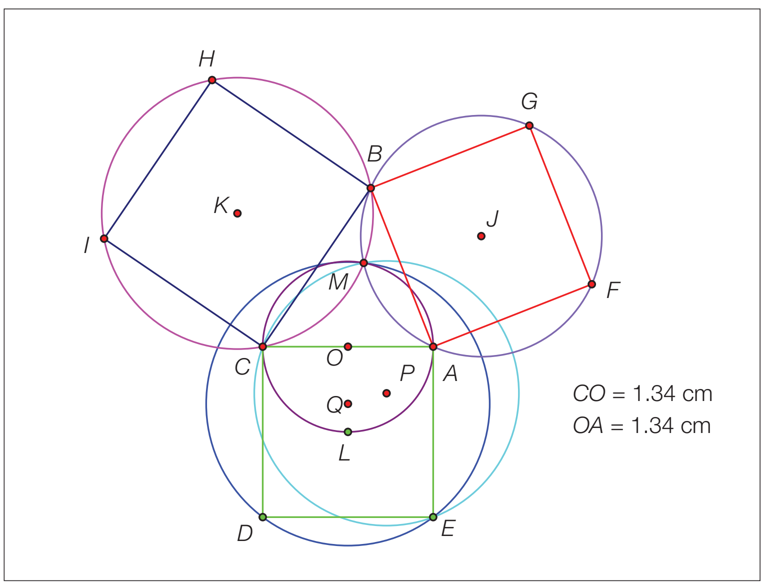


*Figure 12.* Some additional circles going through point M.

patterns. Because I am already familiar with a related pattern, I suggest students to construct the circumcircles of two of the squares (Figure 11). I then challenge the class to construct a circle “related” to the third square and containing point M. As shown in Figure 12, a few students construct the circle going through points M, C and A while others construct circles containing points M, C and E or M, D, and E. As students drag one of the flexible points of the triangle, say B, to study the behavior of the three new circles, some notice that only one circle has a fixed center, the midpoint of the side $\overline{CA}$. The class then typically formulates a conjecture along the following lines:

> Construct squares on the two sides of an arbitrary triangle ABC (Figure 13). Construct the circumcircles of the two squares and the circle with diameter the side of the third square. The three circles are concurrent.
>
> A proof that the class often develops is the following:
>
> Let M be the second point of intersection of the circumcircles of the two squares (Figure 13). Construct segments $\overline{AM}$, $\overline{BM}$, and $\overline{CM}$. By the inscribed angle theorem, $m(\angle AMB) = 270°/2 = 135°$ and $m(\angle BMC) = 270°/2 = 135°$. Hence, $m(\angle CMA) = 360° - (135° + 135°) = 90°$. Therefore, M belongs to the circle with diameter $\overline{CA}$. In other words, the three circles displayed in Figure 13 are concurrent.

The class then continues searching for additional patterns in the Pythagorean configuration, but this is another story and the discoveries should be discussed another time.

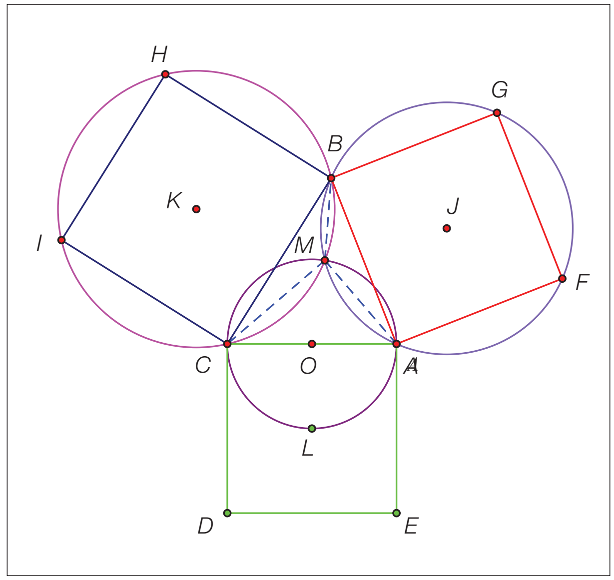

*Figure 13.* Diagram to prove that the three circles are concurrent.

## Discussion

Mathematics is often described as the science of patterns (Devlin, 1996; Steen, 1988, 1990) or as a creative art (Halmos, 1968), to name just a couple of metaphors. As stated by Steen (1990), "Active mathematicians seek patterns wherever they arise" (p. 2). Exploring relationships embedded in the Pythagorean configuration certainly provides students with opportunities to develop their abilities to search for and notice patterns.

One of the goals of education, and mathematics education in particular, should be to "empower our students for life after school…[by helping] them develop genuinely mathematical ways of thinking" (Cuoco, Goldenberg, & Mark, 1996, p. 401). These mathematical ways of thinking, or habits of mind, include experimenting, visualizing, searching for patterns, conjecturing, inventing, and constructing mathematical arguments (Cuoco, Goldenberg, & Mark, 1996).

By constructing segments (Figure 2a and 2b) or circles (Figure 2c) determined by elements of the Pythagorean configuration, students were engaged in the process of experimenting. Some of these experiments led to the discovery of surprising patterns while others did not. By reasoning by analogy and performing thought experiments, as recommended by Cuoco, Goldenberg, and Mark (1996), some students noticed similar patterns. For example, when some students constructed segments $\overline{AH}$, $\overline{BE}$ and $\overline{CF}$ they noticed that segments $\overline{BE}$ and $\overline{CF}$ were congruent and perpendicular (Figure 2b). I then asked the class whether there was another pair of congruent and perpendicular segments. Most of the students in the class responded that segments $\overline{AH}$ (which was not constructed yet) and $\overline{CG}$ had also these properties.

After students performed the experiments (i.e., constructing additional elements) within the Pythagorean configuration, many almost immediately visualized some relationships and patterns (e.g., the congruence of triangles ABC and DIC and the equivalence of triangles AEF and BHG in figure 2a, the congruence or perpendicularity of segments $\overline{BE}$ and $\overline{CF}$ in figure 2b). Certainly, students sometimes formulated a conjecture that seemed to be true for a specific case of a diagram, but analyzing mathematically the situation or testing the conjecture empirically for additional dynamic diagrams usually led to its refutation (e.g., triangles AEF and BHG are congruent). As students gain experience in using dynamic diagrams to visualize relationships and patterns, they may overcome some of the visual obstacles associated with static diagrams. Two of these obstacles are: (1) diagrams are particular examples of a general class of objects and (2) prototypical diagrams may induce irrelevant characteristics of the underlying concept (Yerushalmy & Chazan, 1990).

Once students noticed a pattern, they formulated the corresponding plausible conjecture. As argued by Cuoco, Goldenberg, and Mark (1996), making conjectures is "central to the doing of mathematics" (p. 383). To formulate the conjecture, students used mathematical language to give precise descriptions of its mathematical components, stating clearly what the hypothesis and conclusion were. Students also collected empirical evidence provided by the IGS to support its plausibility.

After students formulated and tested experimentally a conjecture, they were asked to explain why it was true. By constructing a proof, one of the quintessential features of mathematics, students established connections between and among elements of the configuration and previous proved results.

Cuoco, Goldenberg, and Mark (1996) also argue that inventing mathematics should be a habit of mind that teachers should foment in their students. One way to foster students' abilities to invent or create mathematics is through problem posing (Brown & Walter, 1990; Halmos, 1980; Polya, 1973). As students extended the patterns and conjectures embedded within the Pythagorean configuration to diagrams involving non-right triangles, they were applying the process of generalizing, one of the most typical and powerful strategies to create problems, patterns, conjectures, and theorems.

Past and current reforms for school mathematics (e.g., NGA Center & CCSSO, 2010; National Council of Teachers of Mathematics [NCTM], 1989, 2000) call for students to have experiences looking for patterns and discovering, formulating, and generalizing conjectures and theorems. In particular, *Principles and Standards for School Mathematics* (NCTM, 1989) calls for students to have frequent and diverse experiences to "examine patterns and structures to detect regularities; formulate generalizations and conjectures about observed regularities; evaluate conjectures; [and] construct and evaluate mathematical arguments" (p. 262). Certainly, the Pythagorean configuration is a rich source of accessible, surprising, and beautiful geometric patterns. The investigation of these patterns with IGS by students provided them experiences to further develop or reinforce these mathematical habits and processes.

As students investigate patterns in the Pythagorean configuration, they are engaged in a variety of mathematical practices advocated by the *Common Core's Standards for Mathematical Practice* (NGA Center & CCSSO, 2010): Looking for and making use of structure, using appropriate tools strategically, constructing viable arguments and critiquing the reasoning of others, and looking for and expressing regularity in repeated reasoning.

As students actively look for patterns in the Pythagorean configuration, they learn to discern significant patterns from trivial ones; thus further developing their abilities to recognize and understand mathematical structure. As students deepen their understanding of mathematical structure, they will enhance their abilities to look for and make use structure to discover or create new mathematical knowledge.

After students visualize and notice apparent regularities or patterns in the Pythagorean configuration, they formulate an initial conjecture. The use of GSP, or any other type of IGS, enables them to examine a plethora of cases, thus allowing them to confirm, refine, or refute the initial conjecture. Students improve their skills to use IGS strategically to support and deepen their learning recognizing both the insight provided by the software and its limitations. IGS is an appropriate tool that is used strategically to visualize, detect, extend patterns, and formulate the corresponding conjectures.

As students examine several cases to confirm a conjecture related to the Pythagorean configuration, they are engaged in the process of inductively forming a generalization based on data. Students often use counterexamples to refute or refine a conjecture. After students formulate a plausible conjecture, they need to generate arguments to explain, justify, and prove their claims. Their arguments include assumptions, definitions, and previously proved theorems. By sharing their arguments, students provide opportunities to other members of the class to evaluate or respectfully critique them.

The Pythagorean configuration affords students opportunities to look for and express regularity in repeated reasoning. First, the Pythagorean configuration is a rich source of mathematical patterns, discernible regularities whose elements follow a predictable rule or behavior. Second, most of the regularities noticed in the Pythagorean configuration repeat when the configuration involves a non-right triangle. Last, but equally significant, the reasoning used to prove the conjectures about the patterns embedded in the Pythagorean can be extended (repeated) to non-right triangles.

## Concluding Remarks

By investigating patterns in the Pythagorean configuration, my students experienced searching for and noticing patterns. As they formalize their patterns orally or in writing, they may develop their abilities to formulate conjectures and problems. As students develop proofs to justify their conjectures and reformulate them as theorems, they engage in one of the most essential processes of doing mathematics. As students generalize or extend their initial conjectures, problems, and theorems to other similar geometric figures, they further develop their creative abilities. Thus, my students were given opportunities to further develop or reinforce the mathematical habits of mind mentioned above, these mathematical ways of thinking that we would like all students to have. In addition, from a teacher education perspective, my students, future teachers of mathematics, experienced learning mathematics in a way that can help them design tasks with the goal of developing in their students these habits of mind.

By investigating patterns embedded in the Pythagorean configuration, my students were also engaged in a plethora of mathematical practices advocated by the Common Core State Standards Initiative (NGA Center and CCSSO, 2010) and *Principles and Standards for School Mathematics* (NCTM, 2000) for all students to experience and develop. As future teachers, my students experienced learning using an approach that they should embrace as they teach mathematics to their future students so that they (future students) experience and develop these mathematical practices and processes.

As a teacher of mathematics, guiding my students to notice patterns, formulating conjectures and problems, extending or generalizing them, and developing proofs to justify them are some of the most joyful teaching activities.

## References


Brown, S. I., & Walter, M. I. (1990). *The art of problem posing* (2nd edition). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Contreras, J. N. (2004). Exploring nonconvex, crossed, and degenerate polygons. *Mathematics Teacher, 98*(2), 80 – 86.

Contreras, J. N. (2009). Generating problems, conjectures, and theorems with interactive geometry: An environment for fostering mathematical thinking by all students. In A. Flores (Ed.), *Mathematics for every student: Responding to diversity, Grades 9-12* (pp. 71 – 83). Reston, VA: National Council of Teachers of Mathematics.

Cuoco, A., Goldenberg, P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *The Journal of Mathematical Behavior, 15*(4), 375 – 402.

Devlin, K. (1996). *Mathematics: The science of patterns: The search for order in life, mind and the universe.* New York: Holt Paperbacks.

Halmos, P. (1968). Mathematics as a creative art. *American Scientist, 56*(4), 375 – 389.

Halmos, P. R. (1980). The heart of mathematics. *American Mathematical Monthly, 87*(7), 519 – 524.

Hohenwarter, M. (2002). GeoGebra [computer software]. Retrieved from http://www.geogebra.org/cms/en/

Jackiw, N. (2001). The Geometer's Sketchpad [computer software]. Retrieved from http://www.dynamicgeometry.com/

Laborde, J. M., & Bellemain, F. (2005). Cabri 2 [computer software]. Retrieved from http://www.cabri.com/cabri-2-plus.html

Laborde, C., & Laborde, J. (1995). What about a learning environment where Euclidean concepts are manipulated with a mouse? In A. A. diSessa, C. Hoyles, R. Noss, & L. Edwards (Eds.), *Computers for exploratory learning* (pp. 241 – 262). Berlin, Germany: Springer-Verlang.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Retrieved from http://www.corestandards.org

Polya, G. (1973). *How to solve it.* Princeton, NJ: Princeton University Press.

Steen, L. A. (1988). The Science of Patterns. *Science, 240*(29), 611 – 616.

Steen, L. A. (1990). Pattern. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 1 – 10). Washington, DC: National Academic Press.

Yerushalmy, M. & Chazan, D. (1990). Overcoming visual obstacles with the aid of the Supposer. *Educational Studies in Mathematics, 21*(3), 199 – 219.

## APPENDIX A

Construct the height $\overline{EJ}$ corresponding to side $\overline{AF}$ of triangle AEF (Figure A). Construct the height $\overline{CK}$ corresponding to side $\overline{AB}$ of right triangle ABC. By definition of height, $m(\angle AJE) = 90° = m(\angle AKC)$. Because segments $\overline{AE}$ and $\overline{AC}$ are sides of the same square, we have $\overline{AE} \cong \overline{AC}$. Next, $\angle CAK \cong \angle EAJ$ because they have the same complementary angle, $\angle CAJ$. Thus, $\Delta AEJ \cong \Delta ACK$ by the angle angle side congruence criterion. As a consequence, EJ = CK. Finally, because triangles AEF and ACB have congruent bases ($\overline{AF} \cong \overline{AB}$ because they are the sides of the same square) and equal heights, we conclude that they have the same area. A similar argument shows that Area(ΔACB) = Area(ΔBGH).

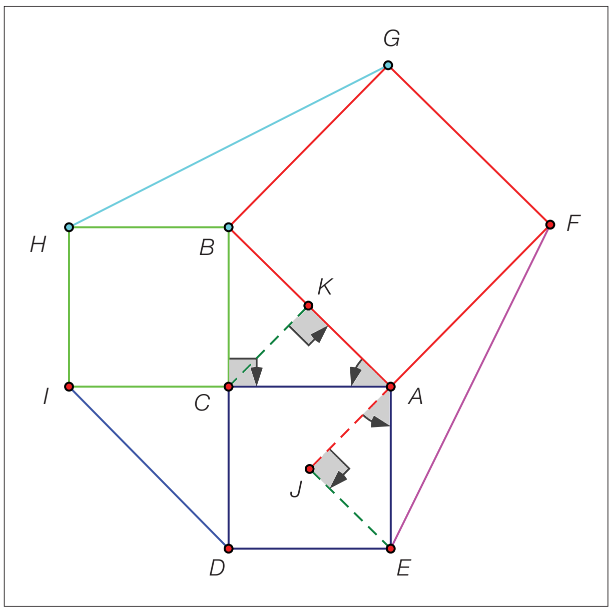


*Figure A.* Diagram to prove that triangles AEF and ACB have the same area.

## APPENDIX B

Construct the diagonals of square AFGB (Figure B). They intersect at J, the center of the square. Since the diagonals of a square are perpendicular, $m(\angle BJA) = 90°$. Now, $m(\angle ACB) + m(\angle BJA) = 90° + 90° = 180°$. Because quadrilateral AJBC has a pair of opposite angles that are supplementary, it is a cyclic quadrilateral. In other words, the circumcircle of ΔABC goes through point J.

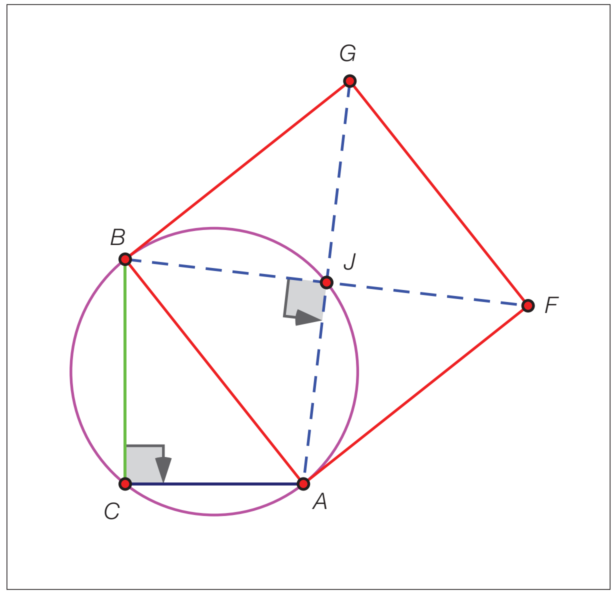


*Figure B.* The circumcircle of right triangle ABC contains J.

## APPENDIX C

Let J be the center of the square constructed on the hypotenuse of a right triangle ABC (Figure C). We want to prove that line $\overleftrightarrow{CJ}$ is the angle bisector of the right angle. First, construct segments $\overline{FJ}$ and $\overline{GJ}$ perpendicular to the corresponding legs of the right triangle. Consequently, ∠BFJ and ∠AGJ are right angles. Since quadrilateral CFJG has three right angles, we infer that it is a rectangle. Thus, ∠FJG is a right angle. Because the diagonals of a square are perpendicular, ∠BJA is also a right angle. Hence, ∠BJF ≅ ∠AJG because each of these angles is the complement of ∠FJA. Since the diagonals of a square are congruent and bisect each other, we have that $\overline{BJ} \cong \overline{AJ}$. Therefore, ΔBFJ ≅ ΔAGJ by the AAS congruence criterion. As a consequence, $\overline{FJ} \cong \overline{GJ}$. In other words, J is equidistant from the sides of ∠BCA and so, it belongs to the angle bisector of ∠BCA.

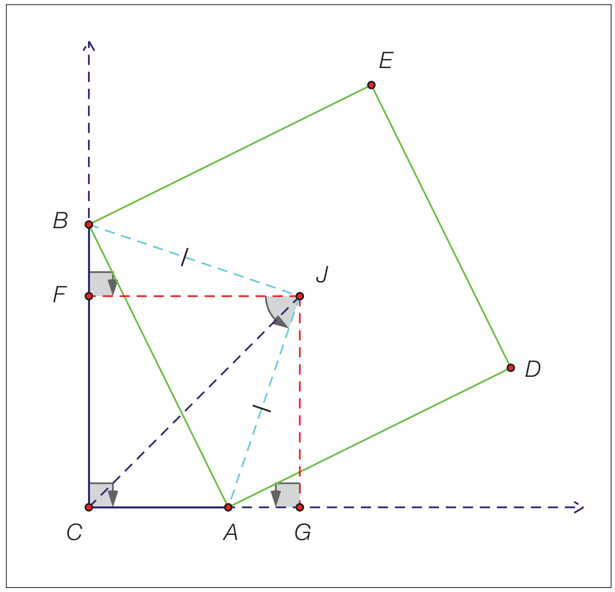


*Figure C.* Ray $\overrightarrow{CJ}$ (not shown) is the angle bisector of the right angle C.