JOURNAL OF MATHEMATICS EDUCATION AT TEACHERS COLLEGE

A Century of Leadership in Mathematics and Its Teaching

Mathematics Teacher Education

TABLE OF CONTENTS

PREFACE

v *Matthew DeGraaf, Teachers College Columbia University*
Beatriz S. Levin, Teachers College Columbia University

ARTICLES

1 **Resources Needed for Addressing Common Core Standards in Mathematics and Language Arts and Next Generation Science Standards**
Margaret B. Cozzens, Rutgers, The State University of New Jersey, Center for Discrete Mathematics and Theoretical Computer Science (DIMACS)

9 **The Effects of Interdisciplinary Practice on Elementary Preservice Teachers' Mathematics Teaching Self-Efficacy and Pedagogy: An Intervention Study of Music-Mathematics Integrated Education**
Song An, University of Texas at El Paso
Daniel Tillman, University of Texas at El Paso
Carlos Paez, Navajo Technical University

25 **Patterns in the Pythagorean Configuration and Some Extensions: The Power of Interactive Geometry Software**
José Contreras, Ball State University

37 **Choosing High-Yield Tasks for the Mathematical Development of Practicing Secondary Teachers**
James A. Mendoza Epperson, University of Texas at Arlington
Kathryn Rhoads, University of Texas at Arlington

45 **Classroom Experience about Cartooning as Assessment in Pre-service Mathematics Content Course**
Hoyun Cho, Capital University, Columbus, Ohio
Carolyn Osborne, Capital University, Columbus, Ohio
Tobie Sanders, Capital University in Columbus, Ohio

JOURNAL OF **MATHEMATICS EDUCATION** AT TEACHERS COLLEGE | SPRING 2015 | VOLUME 6, ISSUE 1

Choosing High-Yield Tasks for the Mathematical Development of Practicing Secondary Teachers

James A. Mendoza Epperson
The University of Texas at Arlington

Kathryn Rhoads
The University of Texas at Arlington

ABSTRACT Many mathematics teacher educators encounter the challenge of creating or choosing mathematical tasks that evoke important mathematical insights and connections yet remain firmly grounded in school mathematics. This challenge increases substantially when trying to meet the needs of practicing secondary mathematics teachers pursuing graduate work in mathematics. This paper articulates characteristics of mathematical tasks that can help teachers develop their mathematical knowledge grounded in school mathematics by drawing on (a) work with practicing secondary mathematics teachers in a mathematics graduate program and (b) relevant research literature on the professional development of mathematics teachers and profound understanding of fundamental mathematics for teaching.

KEYWORDS *profound understanding of school mathematics, high-yield mathematics tasks for teachers, mathematical habits of mind, mathematics teacher education, inservice secondary mathematics teachers, graduate program in mathematics for teaching*

Introduction

Selecting, planning, and implementing mathematical activities for effective instruction require that secondary mathematics teachers possess strong knowledge in mathematics, yet research has shown that mathematical knowledge gained during undergraduate studies in mathematics is not always sufficient for high-quality instruction at the secondary level (e.g., Kahan, Cooper, & Bethea, 2003; Rhoads, 2014). To handle unplanned mathematical events in the classroom, such as answering unexpected mathematical questions from students and evaluating a variety of student solutions, teachers must enact a flexible and deep command of mathematics content (Ball, Thames, & Phelps, 2008; Loucks-Horsley, Love, Stiles, Mundry, & Hewson, 2003). Ma (1999), in her work with elementary teachers, described such mathematical knowledge as *profound understanding of fundamental mathematics*, which is "an understanding of the terrain of fundamental mathematics that is deep, broad, and thorough" (p. 120). Teachers with a profound understanding of school mathematics can connect the topics they are teaching to both more conceptually powerful ideas and ideas with similar or less conceptual power, and they appreciate multiple perspectives of mathematical ideas and can provide mathematical explanations of these perspectives.

Continuing education for practicing secondary teachers can build or strengthen teachers' profound understanding of the mathematics they teach, which supports their preparedness to address the complex mathematical issues that arise in the classroom. Loucks-Horsley et al. (2003) argued, "By becoming a learner of the content, teachers broaden their own understanding and knowledge of the content that they are addressing with their students" (p. 194 – 195). But what types of mathematical experiences can support practicing secondary mathematics teachers in developing the unique mathematical understandings needed for teaching? That is the question we address in this paper.

Description of Our Program

At The University of Texas at Arlington, we work with practicing secondary teachers in a graduate program in mathematics designed specifically for this group. We view the purpose of graduate studies in mathematics as extending undergraduate studies in mathematics, often in an area of specialization. As such, the Master of Arts (M.A.) degree in mathematics extends undergraduate mathematics in the area of specialized content for secondary teaching. We aim to offer teachers opportunities that deepen and connect their mathematical knowledge of high school concepts from an advanced standpoint—opportunities that they have not encountered in their K-12 or undergraduate experiences. For example, the course Concepts and Techniques in Algebra builds on teachers' knowledge of abstract algebra and other undergraduate mathematics for an in-depth exploration of topics from high school algebra, the course Concepts and Techniques in Real Analysis uses an historical approach to the development of analysis to connect teachers' knowledge of real analysis to central ideas in precalculus and calculus, and the course Concepts and Techniques in Problem Solving focuses on tasks that are best resolved using high school mathematics yet elicit advanced mathematical thinking and the development of mathe--matical habits of mind.

Challenging mathematical tasks that are designed to develop profound understanding of high school mathematics are central to the courses in the M.A. program. Drawing from Watson and Sullivan (2008), we define a *task* as a mathematical problem, prompt, or guided exploration that is posed to learners and is "the starting point of mathematical activity" (p. 109). We consider a task to be high yield for teachers if it is grounded in school mathematics with multiple avenues for developing profound understanding of school mathematics and practicing mathematical habits of mind. The tasks we describe in this paper are those in which teachers engage as adult learners in a university setting; they are not necessarily tasks that would be used with secondary students. In addition, the tasks that we use are primarily mathematical with the main goal of enhancing teachers' profound understanding of the mathematics they teach as opposed to their knowledge of mathematics pedagogy (cf. Liljedahl, Chernoff, & Zazkis, 2007).

Over the 15 years since the inception of the M.A. program in Mathematics, we have worked to refine our understanding of high-yield tasks for secondary mathematics teachers by drawing on the research literature, continuously reflecting on the mathematical tasks implemented in our program, collecting data from teachers, and revising tasks accordingly (in a process similar to Liljedahl et al., 2007; see also Loucks-Horsley et al., 2003). Written survey data from this process contributes to our understanding of high-yield tasks, and we share some of this data in this paper.

Characteristics of High-Yield Mathematical Tasks for Secondary Teachers

In our work with hundreds of students in the M.A. program for secondary mathematics teachers, we have identified the following characteristics of high-yield mathematical tasks. In this section we discuss these characteristics generally, and in the following section we illustrate examples of tasks that we have developed according to these guidelines.

Grounded in Secondary School Mathematics and Teaching

Researchers have argued that effective learning experiences for teachers are centered on concepts that are relevant to teachers. Learning advanced mathematics for personal enjoyment is certainly a valuable experience for teachers, but if the content teachers learn is disconnected from teaching, it will not necessarily improve secondary students' learning (Loucks-Horsley et al., 2003). Teachers in our M.A. program have echoed this point in evaluation surveys, commenting on the relevance of specific content. For example, one teacher valued content that was more than "academically stimulating and satisfying;" this teacher appreciated content that directly connected to secondary teaching. Another teacher wrote about their favorite course in the program, saying, "[The course] provided deeper understanding of some of the concepts I teach and many new techniques and details that allowed me to use [this understanding] in my classroom."

One way to connect mathematical tasks to school mathematics is to build on teachers' wonderings about mathematics. For example, when teachers in our program are asked to list their "why" questions, they typically list questions such as "Why isn't $0^0 = 1$? I thought anything raised to the zero power is 1?" or "How do we know when extraneous solutions might arise in solving equations?" When learning mathematics-specific technologies, they ask questions such as "When we use dynamic technology to vary the parameter a in $f(x) = ax$, the original graph of the function seems to rotate. Is the resulting graph a rotation of the original graph?" Tasks can build on these important questions to help strengthen teachers' knowledge of these ideas.

On the other hand, mathematical tasks do not always have to arise from teachers' questions to be relevant. For instance, tasks can also take an answer to a typical high school mathematics problem and use this answer as a starting point for further analysis, including looking at extreme cases of the problem situation, generalizing the problem, or comparing the problem to other tasks (Stanley & Sundström, 2007), and research has shown positive effects of this approach as well (e.g., Bloom, 2007). Whether tasks arise from teacher questions, from answers to typical high school questions, or otherwise, for teachers to understand concepts in new and different ways, it is important that teachers see the explicit connections between the mathematics they are learning and the mathematics they will teach (see also Papick, 2011).

Build Deeper and More Flexible Understandings of Concepts

Researchers have shown that although many secondary teachers have some conceptual understandings of secondary mathematics, such as the familiarity with multiple representations, their capacity to make deep connections may be limited (e.g., Bryan, 1999; Hansson, 2005). For example, although few high school mathematics teachers have difficulty in stating the definition of *function*, teachers are not always aware of how this definition connects to multiple representations of functions. Researchers have reported that many secondary teachers believe that functions are equivalent to algebraic formulas, the terms *function* and *equation* are interchangeable, or graphs of functions are always continuous with no sharp corners (Epperson & Meeks, 2013; Even, 1993; Hitt, 1998; Wilson, 1994).

Mason (2008) argued that part of learning mathematics for teaching is developing *sensitivity* and *awareness* to sometimes subtle but important connections among ideas. To build deeper and more flexible understandings of concepts, tasks must connect new ideas to teachers' existing understandings of mathematics (Hsu, Kysh, Ramage, & Resek, 2007). As Karp (2007) argued, "one of the most important aims of classes on problem solving for mathematics teachers is to show them the all-too-unfamiliar face of nonroutine school-level mathematics" (p. 408). Tasks can build on teachers' existing knowledge of mathematics and provide *cognitive dissonance*—a disagreement between teachers' existing knowledge and the new information being explored. Researchers have argued that this dissonance is a key element in transformative learning experiences for teachers (Loucks-Horsley et al., 2003; Watson & Mason, 2007). These experiences can help teachers to develop what Thompson, Carlson, and Silverman (2007) described as *coherent mathematical meanings.* That is, well-designed tasks can help teachers to form connections among teachers' existing conceptual ideas so that they form a more coherent whole.

Graduates of our M.A. program often cited the opportunity to strengthen their existing knowledge of mathematics as one of the best aspects of the program. One teacher wrote, "[The courses] allowed me to see concepts that I thought I knew well from a different perspective." Another teacher wrote, "I can now provide background as to why things are as they are, how they develop, and why."

Reinforce Mathematical Habits of Mind

As teachers learn mathematics, they also come to understand what it means to do mathematics (Loucks-Horsley et al., 2003). Mathematicians engaging in mathematical activity exercise mathematical habits of mind, which include but are not limited to (a) searching for and explaining patterns, (b) making and checking conjectures, (c) visual thinking, (d) reasoning with multiple representations, (e) inventing mathematics, and (f) providing convincing arguments and proofs (e.g., Cuoco, Goldenberg, & Mark, 1996). These habits (and others similar to this list) are part of guiding standards for school mathematics in the United States, including the Standards for Mathematical Practice in the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and the Process Standards in *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000).

Many teachers have not experienced the teaching or learning of high school mathematics by these processes, and tasks that emphasize mathematical habits of mind can help make teachers aware of the opportunities for mathematical richness that exist in high school mathematics (Watson & Mason, 2007). Loucks-Horsley et al. (2003) argued that by experiencing habits of mind in learning mathematics, "teachers begin to see…mathematics teaching as less a matter of knowledge transfer and more an activity in which knowledge is generated through making sense of or understanding the content" (p. 195).

Examples of Tasks

To illustrate the characteristics described in the previous section, we offer two examples of tasks we have used with practicing secondary teachers in our graduate program.

Transformations of Functions

Using transformations to build new functions from existing functions can generate surprising questions from secondary students when using technology in the classroom or when attempting to relate function transformations to the transformations and notation from geometry. To uncover some of the anticipated surprises, we use a task that requires teachers to use dynamic technology to examine the effect on the graph of replacing $f(x)$ with $f(x) + b$, $b * f(x)$, $f(bx)$, and $f(x + b)$ where b is a positive real number. After experimentation with common functions in school algebra (e.g., $f(x) = x$, $f(x) = x^2$, $f(x) = e^x$, $f(x) = \ln x$, $f(x) = |x|$, $f(x) = 1/x$, $f(x) = \sqrt{x}$, $f(x) = x^n$, $f(x) = \sin x$, $f(x) = \arcsin x$), we pose possible student questions that expose unexpected patterns or outcomes. When these issues arise, it is critically important that teachers be able to effectively use visual thinking or reasoning with multiple representations. For example, in an open-ended exploration of transformations on $f(x) = e^x$, teachers may observe that the graph of $f(x + b)$ looks like the graph of $a * f(x)$ where a is a positive real number. We pose a student question, "Why does the graph of $f(x + b)$ look like both a horizontal shift of the graph of $f(x)$ and a vertical stretch of the graph of $f(x)$?" From here, teachers can use multiple representations, including symbolic notation or numerical thinking with tables, to justify this result.

Similarly, based on their experience with transformations of polynomial functions, teachers may recognize that the graph of $f(x) = 1/x$ does not behave in the way that they expect. Specifically, teachers often predict that multiplying f by a positive real number greater than 1 will lead to a vertical stretch in the graph of the resulting function, but the graph of the resulting function also appears to have stretched horizontally. Teachers typically find this result surprising and may be challenged to readily provide a mathematical explanation. Fluent use of multiple representations leads teachers toward the mathematical explanations for the seemingly simultaneous transformations viewed for $f(x) = 1/x$.

In alignment with the characteristics presented in the previous section, this task **arises from secondary mathematics**. In the secondary curriculum, it is common to give students graphs of functions and ask how a "parent function" of this graph was transformed to obtain the graph (Texas Education Agency, 2012, p. 19). The answer to this question is not straightforward in every case, as teachers discover in this exploration.

In addition, **a deeper understanding and more flexible thinking** about transformations of functions are emphasized, and this task helps to create cognitive dissonance for teachers. The goal with this task is not to introduce new concepts to teachers, but to build on teachers' existing knowledge to develop a deeper understanding of transformations of functions. Most secondary teachers would have little difficulty symbolically verifying a statement such as "If $f(x) = x$ then $f(x + b) = f(x) + b$," but teachers' understanding may not go beyond rules for moving symbols on paper and they may not anticipate the possible questions that students may generate when experimenting with dynamically changing the parameter values. By exploring transformations through multiple representations, the meaning underlying these representations becomes illuminated so that teachers can consider transformations more flexibly.

This task also **reinforces mathematical habits of mind.** As teachers explore transformations of functions, they are asked to make and check conjectures, engage in visual thinking, and reason with multiple representations.

Finite Differences in Data

Another high-yield task we use with teachers explores finite differences in bivariate data—that is, investigating a pattern of growth by equal differences over equal intervals. The idea for this task came from teachers' wonderings about finite differences, a topic that is often included in secondary textbooks but one that most teachers have not had the opportunity to explore conceptually or connect to higher level mathematics.

In high school, students typically explore finite differences using tables of bivariate data and are told that if the first differences over equal intervals in this table are constant, then the data can be modeled with a linear function, if the second differences over equal intervals are constant, then the data can be modeled with a quadratic function, and so on. Students are then taught a procedure for finding a defining expression for a function that models the data. However, few secondary teachers have considered the concepts behind these procedures, and many teachers make incorrect generalizations, such as believing that the data showing growth or decay by equal differences over equal intervals is sufficient for claiming a definitive linear relationship.

To address these issues, we use a task that begins by prompting teachers to observe, conjecture, and prove by induction that the *n*th differences of an *n*th degree polynomial function are constant. Making connections among representations are a key part of this process, as teachers can observe this result in visual, numerical, and symbolic representations.

To follow this justification, we use the following exploration to illustrate that constant *n*th differences in data do not necessarily imply that the data was generated by an n^{th} degree polynomial function. To begin, for h a real number, we define Δ_h to be an operator that gives the difference between $f(x+h)$ and $f(x)$ for a given function f of an argument x. That is, $\Delta_h f(x) = f(x+h) - f(x)$. We use the notation $\Delta_h(\Delta_h f(x)) = \Delta_h^2 f(x)$, and we call Δ_h the *first difference operator*, Δ_h^2 the *second difference operator*, and so on. For the function, $q(x) = x^2 + 3x + \sin(2\pi x)$, teachers create a corresponding table of values in which the x-values are integers over equal intervals and then find $\Delta_1 q(x)$ and $\Delta_1^2 q(x)$ for the values in their table. Because q is not a quadratic function, many teachers are perplexed when they discover that $\Delta_1^2 q(x)$ is constant (see Figure 1). Graphing $y = q(x)$ can help teachers to see why this is the case: The ordered pairs from the table for q lie along a quadratic function. Using dynamic graphing software, teachers can vary the x-values in their ordered pairs and interval lengths to conjecture the conditions that must be met so that the data in the table will yield constant second differences. As seen in Figure 2, teachers can vary the values of x to observe, for example, that for any set of x-values spaced over equal intervals of length 1, the corresponding points $(x, q(x))$ will lie along a quadratic function.

The conjectures that arise from this experimentation with technology require fluent use of multiple representations (e.g., algebraic justification) to support their work. After they engage in the mathematics, teachers are asked to reflect on how they would address finite difference activities with their students based on their work in this task.

x	$q(x)$	$\Delta_1 q(x)$	$\Delta_1^2 q(x)$
– 4	4	– 4	2
– 3	0	– 2	2
– 2	– 2	0	2
– 1	– 2	2	2
0	0	4	2
1	4	6	2
2	10	8	
3	18		

Figure 1. Table of values for $q(x) = x^2 + 3x + \sin(2\pi x)$ with $\Delta_1 q(x)$ and $\Delta_1^2 q(x)$ calculated.

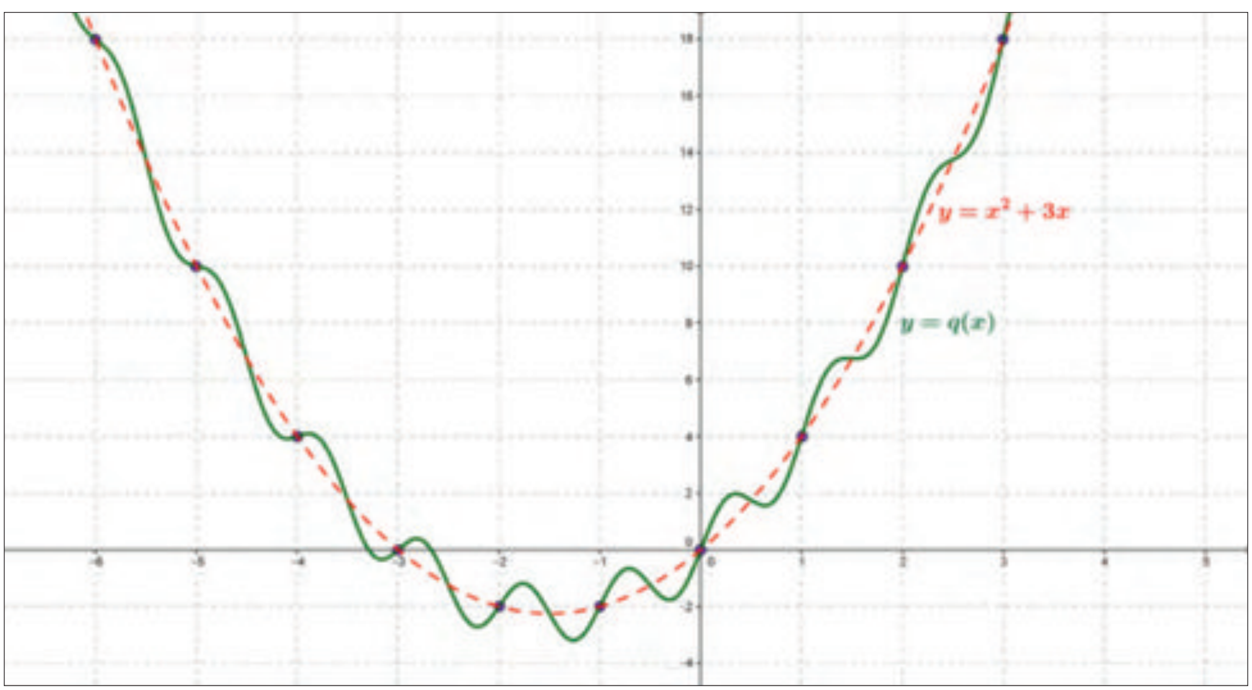

Figure 2. $q(x) = x^2 + 3x + \sin(2\pi x)$ is graphed with a solid curve. The plotted points are $(x, q(x))$ for integer x-values over unit intervals, and these points lie along the dashed curve $y = x^2 + 3x$.

To extend the finite difference concept to higher mathematics, we connect the difference operator to the more general concept of rate of change. For a quadratic function g, teachers are prompted to calculate $\Delta_h^2 g(x)$ for $h = 2$, $h = 1$, and $h = 0.5$. For each value of h, the second differences are constant; however, $\Delta_2^2\ g(x) \neq \Delta_1^2\ g(x) \neq \Delta_{0.5}^2\ g(x)$ (See Figure 3).

The patterns illustrated in Figure 3 are a natural starting point for discussion about rate of change. Teachers can be asked questions such as, "Why does each table have a different value for the second differences?" and "Can you predict the value of the second difference for x-values 0.25 units apart?" Teachers can then be prompted to make connections to the difference quotient and derivative. We find that teachers often do not naturally connect finite differences in secondary mathematics to foundational ideas related to instantaneous rates of change, and after completing this task, teachers convey a renewed appreciation and motivation for investigating finite differences.

In summary, this task exemplifies the characteristics we have outlined in this paper. First, this task **arises from secondary mathematics**, and in particular, this exploration was created in response to a teacher question in one of our courses. Teachers are curious about topics such as finite differences for which they have limited mathematical background. Second, this task is designed to support a **deeper understanding and more flexible thinking** about finite differences. In particular, this task helps to raise teachers' awareness of the inherent limitations of a table of values and the generalizations that can be drawn from it. This task also supports teachers' development of profound understanding of the mathematics they are teaching by offering opportunities to explore the conceptual underpinnings of finite differences and the connections to more advanced mathematical ideas. Third, this task **reinforces mathematical habits of mind** in that it provides opportunities for teachers to observe and use patterns, make conjectures, justify their findings, and use multiple representations to explore the concepts.

Discussion

Because the tasks presented in this paper intentionally emanate from topics directly grounded in school mathematics, aspects of the tasks may appear elementary and basic to mathematicians and mathematics teacher educators. However, their value as tasks for practicing teachers directly corresponds to the depth of reasoning and the questioning that accompanies the tasks. The mathematical thinking and reasoning required presents new mathematical opportunities for teachers typically not encountered in undergraduate studies, and these experiences can enhance teachers' profound understanding of mathematics for secondary teaching. As Ma (1999) calls for elementary teachers to develop a profound understanding of the mathematics they teach, we also argue that secondary teachers need to develop a profound understanding of the mathematics that they teach, with heightened awareness and sensitivity to secondary mathematics concepts. Yet this type of understanding is not something that teachers acquire automatically in their K-12 or undergraduate education. As such, tasks such as those that we present can help teachers to extend their work with mathematics. The tasks take familiar mathematics—such as transformations of functions and finite differences—but require teachers to ponder new associated questions, justify their thinking, and delve deeper into the mathematics that they see as relevant for their teaching.

We argue that tasks are suitable for teachers' exploration if the depth of the mathematical thinking needed to engage in the task is just beyond a teacher's reach. The

x	$g(x)$	$\Delta_2 g(x)$	$\Delta_2^2 g(x)$
– 4	15	– 18	16
– 2	– 3	– 2	16
0	– 5	14	16
2	9	30	16
4	39	46	16
6	85	62	16
8	147	78	
10	225		

x	$g(x)$	$\Delta_1 g(x)$	$\Delta_1^2 g(x)$
– 4	15	– 11	4
– 3	4	– 7	4
– 2	– 3	– 3	4
– 1	– 6	1	4
0	– 5	5	4
1	0	9	4
2	9	13	
3	22		

x	$g(x)$	$\Delta_2 g(x)$	$\Delta_{0.5}^2 g(x)$
– 2	– 3	– 2	1
– 1.5	– 5	– 1	1
– 1	– 6	0	1
– 0.5	– 6	1	1
0	– 5	2	1
0.5	– 3	3	1
1	0	4	
1.5	4		

Figure 3. $g(x) = 2x^2 + 3x - 5$. The first and second differences are calculated for x-intervals of size 2 (first table), 1 (second table), and 0.5 (third table).

term *just beyond* is important and akin to Vygotsky's concept of *zone of proximal development* for children's learning (e.g., Vygotsky, 1930-1934/1978). Challenging existing mathematical ideas promotes learning; however, the mathematical thinking needed to engage in a task should not be so advanced that teachers cannot connect the task to their existing knowledge or their immediate mathematical needs for teaching. The teachers in our program have shared these points as well: In surveys teachers have indicated that they appreciated being challenged by mathematics content that went beyond their undergraduate studies yet was still attainable and connected to their classroom teaching.

On exit surveys taken at the completion of their M.A. degree, teachers indicated high levels of satisfaction with the mathematics they encountered and reported benefits for their students. For example, one teacher wrote,

> Teaching the same algebra curriculum for eight years had really stifled my passion for mathematics. [This] program has been absolutely reinvigorating. The [program's] curriculum is not only incredibly broad, covering everything from fundamental arithmetic to the historical challenges that shaped our modern understanding of calculus, but is presented with a depth and clarity that has given me numerous insights into how I can better challenge my students to think more critically and more readily prepare them to integrate algebraic concepts into their higher level math courses.

More than 100 years ago, mathematicians and mathematics teacher educators called for developing students' and teachers' mathematical habits of mind and promoted the notion that deep and connected mathematical knowledge is important for mathematics teachers (American Mathematical Society, 1899/1970), and this call has been especially pronounced in the last 40 years (Schoenfeld, 2001). The challenge lies in provoking mathematical habits of mind and deep connected mathematical knowledge while remaining tied to school mathematics. There is a dire need for tasks—especially those suitable for use in graduate programs—for secondary mathematics teachers that faithfully begin in school mathematics and unpack sophisticated mathematical ideas to help teachers develop a profound understanding of mathematics for teaching rather than tasks that begin at a high level of abstraction and then backmap to school mathematics. There is also an acute need for tasks that directly focus on the "uncanny unpacking" (Ball et al., 2008) of *secondary* school mathematics, as there are many more examples originating in elementary school mathematics (Stacey, 2008). Future research on the impact of high-yield tasks on secondary mathematics teachers' mathematical understandings and further development of high-yield tasks remain important. The characteristics articulated in this paper provide a starting point and scaffold thinking about choosing and creating high-yield tasks to meet the need for more examples that specifically address profound understanding of secondary mathematics for teaching.

References

American Mathematical Society (1899/1970). Report of the committee of the Chicago Section of the American Mathematical Society. In J. K. Bidwell & R. G. Clason (Eds.), *Readings in the history of mathematics education* (pp. 195 – 209). Washington D.C.: NCTM.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389 – 407.

Bloom, I. (2007). Extended analyses: Promoting mathematical inquiry with preservice mathematics teachers. *Journal of Mathematics Teacher Education, 10*(4-6), 399 – 403.

Bryan, T. J. (1999). The conceptual knowledge of preservice secondary mathematics teachers: How well do they know the subject matter they will teach? *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal, 1*, 1 – 12.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behavior, 15*(4), 375 – 402.

Epperson, J. A. M., & Meeks, V. (2013, November). *Probing inservice teachers' understanding of functions vs. equations.* Paper presented at the School Science and Mathematics Association (SSMA) 2013 Annual Convention, San Antonio, TX.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education, 24*(2), 94 – 116.

Hansson, O. (2005). Preservice teachers' view on $y = x + 5$ and $y = \pi x^2$ expressed through the utilization of concept maps: A study of the concept of function. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th conference of the International Group for the Psychology of Mathematics Education*, *Vol 3* (pp. 97 – 104). Melbourne: PME.

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior, 17*(1), 123 – 134.

Hsu, E., Kysh, J., Ramage, K., & Resek, D. (2007). Seeking big ideas in algebra: The evolution of a task. *Journal of Mathematics Teacher Education, 10*(3), 325 – 332.

Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education, 6*(4-6), 223 – 252.

Karp, A. (2007). "Once more about the quadratic trinomial…": On the formation of methodological skills. *Journal of Mathematics Teacher Education, 10*(4-6), 405 – 414.

Liljedahl, P., Chernoff, E., & Zazkis, R. (2007). Interweaving mathematics and pedagogy in task design: A tale of one task. *Journal of Mathematics Teacher Education, 10*(4-6), 239 – 249.

Loucks-Horsley, S., Love, N., Stiles, K. E., Mundry, S., & Hewson, P. W. (2003). *Designing professional development for teachers of science and mathematics* (2nd ed.). Thousand Oaks, CA: Corwin Press, Inc.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Mason, J. (2008). PCK and beyond. In P. Sullivan & T. Wood (Eds.), *The international handbook of mathematics teacher education: Vol 1. Knowledge and beliefs in mathematics teaching and teaching development* (pp. 301 – 322). Rotterdam, the Netherlands: Sense.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics* [Online version]. Retrieved from http://www.nctm.org/standards/content.aspx?id=26792

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Retrieved from http://www.corestandards.org

Papick, I. J. (2011). Strengthening the mathematical content knowledge of middle and secondary mathematics teachers. *Notices of the American Mathematical Society, 58*(3), 389 – 392.

Rhoads, K. E. (2014). *High school mathematics teachers' use of beliefs and knowledge in high-quality instruction* (Doctoral dissertation). Retrieved from https://rucore.libraries.rutgers.edu/

Schoenfeld, A. H. (2001). Mathematics education in the twentieth century. In L. Corno (Ed.), *Education across a century: the centennial volume* (pp. 239 – 278). Chicago, IL: University of Chicago Press.

Stacey, K. (2008). Mathematics for secondary teaching: Four components of discipline knowledge for a changing teacher workforce. In P. Sullivan & T. Wood (Eds.), *The international handbook of mathematics teacher education: Vol. 1 Knowledge and beliefs in mathematics teaching and teaching development* (pp. 87 – 113). Rotterdam, The Netherlands: Sense.

Stanley, D., & Sundström, M. (2007). Extended analyses: Finding deep structure in standard high school mathematics. *Journal of Mathematics Teacher Education, 10*(4-6), 391 – 397.

Texas Education Agency. (2012). *Texas essential knowledge and skills: Chapter 111c mathematics* [Online version], Retrieved from http://ritter.tea.state.tx.us/rules/tac/chapter111/ch111c.pdf

Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education, 10*(4-6), 415 – 432.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes.* (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman., Eds.) (A. R. Luria, M. Lopez-Morillas, & M. Cole [with J. V. Wertsch], Trans.) Cambridge, MA: Harvard University Press. (Original manuscripts [ca. 1930 – 1934])

Watson, A., & Mason, J. (2007). Taken-as-shared: A review of common assumptions about mathematical tasks in teacher education. *Journal of Mathematics Teacher Education, 10*(4), 205 – 215.

Watson, A., & Sullivan, P. (2008). Teachers learning about tasks and lessons. In D. Tirosh & T. Wood (Eds.), *The international handbook of mathematics teacher education: Vol 2. Tools and processes in mathematics teacher education* (pp. 109 – 134). Rotterdam, the Netherlands: Sense.

Wilson, M. R. (1994). One preservice secondary teacher's understanding of function: The impact of a course integrating mathematical content and pedagogy. *Journal for Research in Mathematics Education, 25*(4), 346 – 370.