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## Abstract Algebra to Secondary School Algebra: Building Bridges

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**ABSTRACT** The authors have experience with secondary mathematics teacher candidates struggling to make connections between the theoretical abstract algebra course they take as college students and the algebra they will be teaching in secondary schools. As a mathematician and a mathematics educator, the authors collaborated to create and implement a three-semester project to help secondary mathematics teacher candidates inform their teaching using abstract algebra.

**KEYWORDS** *abstract algebra, systems of linear equations, secondary mathematics teacher candidates, collaboration*

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### Introduction

“Why is the product of two negative integers a positive integer?” “Why is the elimination method of solving systems of linear equations a legitimate solution method?” “When am I ever going to need this?” How many of these questions have you encountered in your mathematics classroom? These questions are typical questions in our undergraduate abstract algebra course. Many of the secondary mathematics teacher candidates we encounter believe that abstract algebra is the least relevant course in their major, while we assert that abstract algebra has the greatest relevance to the school mathematics they will be teaching. Mathematical structures form the basis of our number system and provide the underlying foundation for answering the first two questions posed. In our experience, the highly symbolic nature of undergraduate abstract algebra can make it difficult for secondary mathematics teacher candidates to see connections between this classic “definition-theorem-proof” course and the mathematics that they will be teaching at the secondary level.

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### The Beginning of a Collaboration

As a mathematician and a mathematics educator, we were determined to break through our secondary mathematics teacher candidates’ views regarding abstract algebra. It was important for us to help these budding educators use abstract algebra to inform their teaching. The desire to help future educators inform their teaching using abstract algebra is not unique to our institution. According to *The Mathematical Education of Teachers II* (Conference Board of the mathematical Sciences [CBMS], 2012) report, newly graduated secondary mathematics teachers experience a content “jolt moving from the mathematics major to teaching high school” (p.53). This report suggests that secondary mathematics teacher candidates should have course experiences that are active in “examining connections between middle grades and high school mathematics as well as those between high school and college” (CBMS, 2012, p. 54). In addition, Wu states that

This necessity that math teachers actually know the mathematics they teach sheds light, in particular, on why we want all high school teachers to know some abstract algebra...real progress in teacher education will require both the education and the mathematics communities to collaborate very closely" (2011b, p. 380).

Further, Wu states "What is needed to bridge this gulf is the concept of customizing abstract mathematics for use in the school classroom" (Wu, 2011b, p. 377).

Our goal is to help our secondary mathematics teacher candidates become more aware of the connections between the algebraic structures they study in college abstract algebra and the use of those algebraic structures in teaching secondary mathematics. By helping them lift this veil, we envisioned them discovering that making these connections fosters a richer and deeper experience in mathematics content knowledge.

Ball and colleagues are leaders in researching categories of mathematical knowledge for teaching. One of these categories is specialized content knowledge (Ball, Thames, & Phelps, 2008). According to their work, specialized content knowledge is the mathematics specific to the field of teaching, as opposed to other fields such as the mathematics needed for engineering (Ball, Hill, & Bass, 2005). For example, teachers use their specialized content knowledge of mathematics (such as functions and/or transformations) to "scrutinize, interpret, correct, and extend" (Ball, Hill, & Bass, 2005, p. 17) the developing mathematics of their students. In particular, the work of Gilbert and Coomes in the Mathematics Content Collaboration Communities focuses on the mathematics knowledge needed for teaching secondary mathematics. They conclude that "there are fundamental differences between knowing mathematics and teaching mathematics" (Gilbert & Coomes, 2010, p. 423).

To transform our goal into an action plan, we designed a project that is a sequential three-part, three-semester experience originating in an abstract algebra course and culminating in the semester during student teaching and seminar. Part 1 was implemented at the end of an abstract algebra course. The purpose of Part 1 was to study a high school algebra topic from the perspective of abstract algebraic structures (in particular, systems of linear equations). Part 2 was implemented in a practicum course—a course to prepare students for student teaching. This course is typically taken in the semester immediately following abstract algebra. In Part 2, the teacher candidates revisit the assignment from Part

1 and create a lesson plan from the chosen topic. Part 3 occurred during student teaching and seminar the next semester. The purpose of Part 3 was to have the student teachers reflect upon their experiences in Parts 1 and 2 and upon their ability to recognize connections between abstract algebra at the college level and at the school level.

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## Part 1: Initial Connections in Abstract Algebra

Part 1 occurred in the last month of the semester in which abstract algebra was taken. At our institution, the population in abstract algebra consists of mathematics majors and secondary education mathematics majors. At the time students are assigned this project, they have been exposed to theory of groups, rings, and fields typical to what is seen in a standard undergraduate abstract algebra course. Each student is assigned a different system of two linear equations with two unknowns, where each system has a unique solution. In secondary schools, students are presented with a variety of methods to find solutions to systems of linear equations. Among the variety of methods available, two common symbolic procedures are presented: substitution and elimination. The goal of the Part 1 task is to solve the given system using each method, with only information culled from abstract algebra. Each student in the abstract algebra class was assigned this project, and had to do the following:

1. State the algebraic structure (group, ring, integral domain, field) in which the system is being solved. They must use the most appropriate fit. For example, if a student chose the field of complex numbers, but all known and unknown numbers are real, then the better fit would be the field of real numbers.
2. Create a list of all theorems, homework problems, and/or class notes that they intend to cite in their solution process. With theorems listed, the student is required to note where the theorem is found in the book, in the homework problems, or class notes. If they cannot find the fact, the student has to first prove it in order to use it.
3. Justify each step in the solution process. The algebraic fact being used has to be cited from the list above in step 2.

For example, if a student had to solve for  $(x,y)$  that would satisfy both equations:  $2x + 3y = 8$  and  $-4x + y = 2$ , in step 1, the student would choose to work in the field of rational numbers. In step 2, that student would list the

axioms that define a field (L. Gilbert & J. Gilbert, 2009, p. 271), and what numbers in the set of rational numbers represent the additive and multiplicative identities. Further, the student would need to cite results from early in the semester that addition and multiplication within the set of rational numbers are well-defined, as well as properties of equality (e.g., equality is an equivalence relation) that will help them get through their solution process. As the student works through the solution process, no step can be made unless it has an appropriate justification from abstract algebra

Using as an example the system of linear equations stated previously, if we were to start to solve for  $(x,y)$  using substitution, we could start by solving for  $y$  in the second equation.

Since  $4x = 4x$  (reflexive property of equality) and addition is well defined, we know that  $4x + (-4x + y) = 4x + 2$ . By the associative property of addition, we have  $[4x + (-4x)] + y = 4x + 2$ . By the right distributive property in the field of rational numbers, we conclude  $[4 + (-4)]x + y = 4x + 2$ .

Since  $-4$  is the additive inverse of  $4$ , we have  $0x + y = 4x + 2$ . By the zero product theorem (L. Gilbert & J. Gilbert, 2009, p. 264) we know  $0x = 0$  and so  $0x + y = 4x + 2$  can be equivalently written as  $0 + y = 4x + 2$ .

Since  $0$  is the additive identity in the set of rational numbers,  $0 + y = y$ , and so our equation can now be simplified to  $y = 4x + 2$ . Using the reflexive property of equality (e.g.,  $3 = 3$ ) and the well-defined property of multiplication, we may conclude  $3y = 3(4x + 2)$ —we cannot use the word “substitute,” we must use the proper language from formal algebraic structures.

Similarly, by the reflexivity of equality and the well-defined property of addition,  $2x + 3(4x + 2) = 2x + 3y$ . The transitive property of equality allows us to conclude that  $2x + 3(4x + 2) = 8$ .

We would then continue with the appropriate (abstract algebra) justifications to obtain that  $x = 1/7$ , and so  $y = 18/7$ . Once the demonstration of the solution process using the substitution method is complete, we would start over, this time demonstrating the solution process using the elimination method.

A professionally written product was expected from the student, and was graded by the abstract algebra instructor (the mathematician in this article) as part of the course grade. Furthermore, secondary mathematics teacher candidates were reminded to keep their graded project for use as a reference in their practicum course.

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## Part 2: Using the Abstract Algebra Project to Create a Lesson Plan

Practicum is a course required for secondary mathematics teacher candidates during the semester prior to student teaching. Students examine principles, methods, content, and curriculum in mathematics, and they prepare to implement standards-based lessons that engage all learners. It is important to note that all students are required to successfully complete abstract algebra as a prerequisite to entering the practicum course. Since not all the practicum students were in the same section and semester of abstract algebra, we had a variety of backgrounds to work with. The Practicum students were provided with a mini-review of abstract algebra, and were encouraged to meet with the mathematician privately if needed.

We created two components to Part 2: one component was an assignment similar to that given in the abstract algebra course, and the second component was to create a lesson plan on solving systems of linear equations appropriate for students in a school algebra class based on a newly-assigned system of linear equations. For those who were not exposed to Part 1, a system of linear equations with one solution was assigned for the first component of Part 2. Those who were exposed to Part 1 (had already solved a system of linear equations that had one solution), were assigned a system of linear equations with either no solution or infinitely many solutions. Students had access to the textbook, former class notes, and the mini-review in order to obtain their list of facts to complete the steps for the first component. All students had to solve their given system based on an abstract algebra perspective as described in Part 1.

To assist in the lesson plan component of Part 2, we studied the *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) and directed students to focus on the conceptual category of algebra. Additionally, as outlined in the CCSSM, the need for coherent and correct use of mathematical language and procedures in teaching and learning was emphasized.

To create the lesson plan, students were expected to use the work from the first component of Part 2 and their study of the CCSSM around the topic of solving systems of linear equations. Each teacher candidate had to grapple with how to begin to bridge a solid connection between the algebraic structures in college abstract algebra and school algebra. To assist them, we had them search

the CCSSM K-9 to find the instances in which algebraic structures first appear. For example, the Commutative and Associative Properties of whole number addition appear in Grade 1 (NGA & CCSSO, 2010, p. 15); the Distributive Property of whole number multiplication over addition appears in Grade 3 (NGA & CCSSO, 2010, p. 23). Recognizing that explaining “why” to a high school student may be challenging in light of their prior experience with mathematics, we wanted the teacher candidates to consider how they would guide their students. We emphasized that while we must be precise in the use of mathematical language, we want to avoid rote memorization of rules and procedures. In addition, we had teacher candidates think about how to help high school students internalize certain facts without just pure memorization. For example, as educators we know that the product of two negative integers is a positive integer; that is a consequence of the structure learned in college abstract algebra. Many high school students cannot understand this information at such an abstract level, but still require an appropriate explanation—that can be found on page 406 in Wu’s *Understanding Numbers in Elementary School Mathematics* (2011a). Rather than merely relying on the rote memorization of facts, teacher candidates were expected to facilitate student learning of mathematics beyond such a surface level by providing experiences, such as guided discovery activities, that encourage the construction of meaning for the mathematics being taught.

For the lesson plan, we encouraged teacher candidates to be as creative as possible, especially in the launch to “hook” students into the lesson. We had them think deeply about devising a context or scenario for their respective system of linear equations. Lessons were presented using the proceeding format. Each teacher candidate presented their lesson as if they were a high school algebra teacher while one classmate took on the role of peer observer and the remaining classmates took on the roles of high school algebra students. The two authors (a mathematician and a mathematics educator) were faculty observers. Each role is defined as follows: “Student’s” role: explore the content of the lesson as learners of mathematics. “Observer” role: fill out a rubric which includes a narrative section addressing questions such as: What are the mathematical goals of the lesson? What pivotal questions or activities does the teacher use to move the mathematics forward? What questions do not seem to move the mathematics forward? The faculty observers helped each teacher candidate identify strengths and areas for improvement in the abstract al-

gebra component, lesson plan, and lesson presentation.

Immediately following the lesson presentation, each teacher candidate had a debriefing session with all who were at the presentation. The presenter was asked to discuss how the abstract algebra component informed the lesson. This component was submitted the day of the lesson, and the lesson plan was submitted later with a post-presentation reflection. Each teacher candidate also met privately with each faculty observer for additional feedback.

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### Part 3: Connections in Student Teaching

Part 3 was implemented during student teaching. One of the courses assigned to each student teacher was an algebra class. We were interested in seeing what connections made in the previous two semesters carried over into the student teaching experience. When approximately 75% of the semester was completed, teacher candidates were asked to use one two-hour seminar to respond to two reflection questions during one of their weekly student teaching seminar meetings:

1. During your student teaching experience, how have you used your knowledge of algebraic structures (use abstract algebra course, practicum project, student teaching experience as reference points) to guide your teaching and/or the learning of your students? Provide specific examples.
2. During your student teaching experience, is there an example(s) where, in hindsight, you could have used your knowledge of algebraic structures to guide your teaching and/or the learning of your students? Provide specific examples.

Our student teachers were informed that thoughtful responses were considered important not only to display growth as a beginning mathematics educator starting to see critical connections between college abstract algebra and school algebra, but also were central to how we can determine the impact of our project on their growth, and how to best assist future teacher candidates.

Prior to analyzing the student teachers’ written comments to these questions, we had two over-arching project queries to which we hoped their responses would provide insight. These two queries were: Is the nature of the project assignment during Practicum sufficient to enable them to see connections between algebraic structures and school mathematics as they advanced to student teaching; and if made, do the connections inform their teaching during the student teaching semester?

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## Findings

Responses indicated that the student teachers did make connections between algebraic structures and school mathematics, and those connections did inform their teaching. Some of their comments discuss instances of occurrence of various properties (e.g., commutative property of addition) and operation definitions (e.g., defining subtraction). From their study of algebraic fields, student teachers knew there are only two operations: multiplication and addition. Subtraction is defined only as a notation, that is, something like  $2 - 3$  is a notation for  $2$  plus the additive inverse of  $3$ . One student teacher commented, "It became quite clear to me how embedded in the students' understanding [were] a shallow understanding of numerical operations and properties." Another student teacher stated, "Most students would write  $y = 4 - 3x$  instead of  $y = -3x + 4$ . We would then talk about how the two were equivalent." A similar remark made by a different student teacher indicated:

*a student gave an answer of  $y = 2 - 3x$  [while] another group said that the answer was 'wrong.' They told me the 'correct' answer is  $y = -3x + 2$ . I saw this as a perfect teaching moment. I wrote both answers on the board and asked the class how these two answers were different. [Upon further discussion] they told me both answers were actually the same. When I asked why, they told me it was the commutative property and that both answers are actually the same. This was just one of the many great discussions of how important algebraic structures are in the classroom.*

For instances of issues with the distributive property, a student teacher noted that she made a conscious decision to change her practice for the future: "classes had just learned to 'FOIL' although explained [this was] a usage of the distributive property, [only] 'foiling' stuck. I did not discourage the students from using this term as I now wished I had. Emphasizing the distributive property is a change I will make in future teaching."

Properties and operations with integers was another commonly cited topic. One student teacher noted that using algebraic structures helped her enhance the number sense of her students. As an example, she cited an instance in which she reviewed, or in some cases intro-

duced, the additive identity to explain the concept of zero pairs and how that seemed to have a positive impact on her students' work with integers. Along this avenue, another student teacher observed, "Some grasp that we 'subtract  $x$  from both sides' (in the equation  $y + x = c$ ) not to 'get  $y$  by itself' per se." She said she felt that some students don't really understand that we subtract  $x$  from both sides of an equation so that on one side of the equation we have  $x + (-x)$  'adds to zero.' This student teacher also pointed out that "both identities (additive and multiplicative) appeared most often in my classroom." Other comments seemed to point attention to the use of correct terminology and precision of language rather than the overt use of informal vocabulary that a budding teacher may fall prey to use. In our opinion, comments like these from our student teachers show a heightened awareness of going beyond surface treatment of these topics and concepts as they appear in the classroom. We believe there is a reflective nature to some of the comments suggesting that a student teacher would approach a topic differently in the future.

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## Conclusion

From the written responses we received, we felt that this three-semester project afforded secondary mathematics teacher candidates the opportunity to delve deeper into a common topic in school mathematics, and begin to make critical connections from their college algebraic structures to the school algebra they will be teaching. We found that our experience with this group of teacher candidates shows promise to expand upon this project for future students. We plan on replicating this project's activities into a more formal study to assist teacher candidates in making critical connections between the structures in abstract algebra and school mathematics topics.

This project represents the first endeavor in the history of our department in which a mathematician and mathematics educator collaborated to bridge college abstract algebra and school mathematics as teacher candidates transition from college students to student teachers. Furthermore, we are looking to expand the basic structure of this project to draw connections between teaching secondary mathematics and other college courses.

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## References

- Ball, D., Hill, H., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 29(1), 14 – 17, 20 – 22, 43 – 46.
- Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389 – 407.
- Conference Board of the Mathematical Sciences. (2012). The Mathematical Education of Teachers II. *Issues in Mathematics Education* 17. Retrieved from <http://cbmsweb.org/MET2/met2.pdf>.
- Gilbert, M., & Coomes, J. (2010). What mathematics do high school teachers need to know? *Mathematics Teacher*, 103(6), 418 – 423.
- Gilbert, L., & Gilbert, J. (2009). *Elements of modern algebra* (7th ed.). Belmont, CA: Brooks/Cole.
- National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards*. Retrieved from <http://www.corestandards.org>.
- Wu, H. (2011a). *Understanding numbers in elementary school mathematics*. Providence, RI: American Mathematical Society.
- Wu, H. (2011b). The mis-education of mathematics teachers. *Notices of the AMS* 58(3), 372 – 384.