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A Measurement Activity to Encourage Exploration of Calculus Concepts

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ABSTRACT This article describes an activity that incorporates measurement in order to lead students to discover and explore fundamental concepts of calculus. Students are provided with an experientially real starting point involving measurement and are encouraged to engage in mathematical discussions in a low-stakes environment. I describe the activity as it unfolded in my classroom, while indicating moments at which my students exhibited certain Standards for Mathematical Practice (National Governors Association Center for Best Practices & Council of Chief School Officers, 2010) and Process Standards (National Council of Teachers of Mathematics, 2000).

KEYWORDS *calculus, measurement, exploration, activity*

Both the Common Core State Standards Initiative (National Governors Association Center [NGA] & Council of Chief State School Officers [CCSSO], 2010) and the National Council of Teachers of Mathematics (NCTM, 2000) suggest a need for students at all grade levels to develop proficiency in measurement. Incidentally, some standards documents indicate a connection between measurement and certain calculus concepts. For instance, the NCTM's *Principles and Standards for School mathematics* includes expectations for high school students to be able to "apply informal concepts of successive approximation, upper and lower bounds, and limit in measurement situations" (NCTM, 2000, p. 320). According to the *Common Core State Standards for mathematics* (CCSSM), in transitioning from middle grades to high school, students will begin to extend their use of measurement from familiar contexts such as measuring length, area, and volume to new contexts involving derived quantities and rates (NGA & CCSSO, 2010). Throughout high school, students should become more prepared for the transition from the measurement of fixed quantities to the measurement of changing quantities, a skill needed in the study of calculus.

The authors of *Focus in High School Mathematics: Reasoning and Sense Making* demonstrate the importance of reasoning and sense making in mathematics instruction. They define *reasoning* as "the process of drawing conclusions on the basis of evidence or stated assumptions" (NCTM, 2009, p. 4) and *sense making* as "developing understanding of a situation, context, or concept by connecting it with existing knowledge" (NCTM, 2009, p. 4). The measurement activity described in this article takes the familiar context of measurement as a starting point for instructional activity, thus allowing students to use their existing knowledge of measurement to *make sense* of calculus concepts associated with the measured quantities. Through the process of making sense of these concepts, students should be encouraged to make conjectures and to use the familiar context to *reason* about the new ideas they encounter. From this perspective it is reasonable to believe that a hands-on activity involving measurement could support students in developing their informal understandings of calculus concepts prior to encountering them formally in a calculus course.

This measurement activity is ideal for students at either the high school or college level in courses ranging from Algebra I to Pre-Calculus or even Calculus. It involves an exploration of the relationship between the volume of water in a container, the water level in the container, and the shape of the container. Students are presented with the open-ended task of determining how "fast" the water level is changing in a container as water is being poured into it. The activity is loosely based on a classic problem in calculus and differential equations in which liquid is continuously drained from or added to a container at a given rate (e.g., Stewart, 2008, p. 624 – 625). Calculus would be needed to solve the problem in the case of *continuously* pouring the water into the container, but this activity will require students to take a *discrete* approach to the problem by using fixed measured quantities as units.

The activity is accessible to students who are not yet ready for the study of calculus while offering the potential for students to explore many calculus concepts, including rate of change, the shape of the graph of a function (concavity), approximation, and accumulation. The activity embodies a particular cluster (HSF.IF.B) of the Interpreting Functions domain in the CCSSM Content Standards, which includes:

HSF.IF.B.4.

> For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts, intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*

HSF.IF.B.6.

> Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. (2010, p. 69)

In order to ensure that important content was covered in the classroom, the measurement activity was developed with this content domain in mind. Ideally, students would be free to pursue any mathematical questions that arose, but having content-related goals helped me to provide some structure and direction when students seemed confused or stopped asking questions. These content standards are evident in calculus problems concerned with analyzing the behavior of functions (finding intervals where a function is increasing/decreasing, determining intervals of concavity, finding relative maxima and minima, determining end behavior, and interpreting rates of change). Consequently, this activity takes measurement as a starting point to encourage students to take part in the process of doing mathematics in a low-stakes environment that has the potential to introduce students to concepts of calculus prior to encountering them formally in a calculus course.

Background for Developing the Activity

Creating an environment in which younger students can discover calculus concepts is not a new idea. The SimCalc Project (Kaput & Roschelle, 2013) represents one of the most significant efforts to introduce younger students (mostly in middle grades) to calculus concepts. This approach used computer simulations to develop students' understandings of the underlying calculus concepts including rates of change, approximation, and accumulation. This project supports the idea of introducing students to calculus concepts at an earlier stage, but I chose to diverge from the use of computer simulations for two reasons. First, a pre-planned computer simulation activity restricts students' freedom to pursue various learning paths as they arise. Second, access to a sufficient number of computers for all students to engage in the simulations is not necessarily guaranteed, and this activity is meant to be accessible in all classroom environments.

Driskell and Malagon (2013) presented an activity intended to engage students in measurement on the first day of a college calculus course. The activity used zip lines to introduce students to the relationship between average and instantaneous velocity. An important characteristic of this activity is that students were able to record measurements of distance and time more accurately than would be possible with simple free fall and other forms of motion typically used to demonstrate velocity as a rate of change. This activity informed the development of my own in that it also allowed students to exercise some control over the task in order to collect accurate measurements. Landers (2013) described an activity in which students used measurement to develop an understanding of trigonometric functions and their graphs. During the implementation of Landers's (2013) activity, some students abandoned the practice of measuring angles and distances after making observations about the symmetry of the circle. The instructor then pointed out this observation to the rest of the class, shifting the focus from direct measurement to practical reasoning. The significance for the development of my own activity lies in the potential of measurement activities to act as a starting point from which student observations can generate a wide range of possible learning trajectories.

An activity of this nature parallels the open-ended problems as described in Wu (1994) and Varygiannes (2013). These problems are characterized by a certain ambiguity that allows for the potential of multiple solution paths. The main benefit of open-ended problems is that they can be more accessible than most traditional problems. Moreover, the stakes are lower since there is not necessarily a “right” answer. Wu provides three examples of open-ended problems that would require advanced concepts such as the isoperimetric inequality, the Chinese Remainder Theorem, and optimization of functions of several variables, but are presented in a context that is accessible to students in middle or high school. The purpose of the measurement activity is essentially the same—to make complex mathematical ideas accessible to more students in a low-stakes environment.

The development of this measurement activity was guided by these activities as well as the perspectives on measurement outlined in the standards documents mentioned previously. First, the measurement activity is designed to be practical in the sense that it can be easily replicated by other teachers; so, the activity only requires common materials. It is also important that students are able to take accurate measurements so that they can be confident in their data as a representation of the physical situation. Lastly, the instructor should try to take a laissez-faire approach, allowing students to pursue the mathematical work in their own way rather than showing them how it should be done.

The Measurement Activity

During this activity, students explore the relationship between the volume of water in a container, the water level, and the shape of the container (cross-sections). The activity is described according to how it unfolded in the classroom, which consisted of 8 students in a supplementary program for first-generation college-bound high school students in New York, NY. Students in this class were enrolled in various mathematics classes at their respective high schools, including Algebra I, Geometry, and Algebra II/Trigonometry. The class met weekly on Saturday, and the instructional session described in this article lasted 90 minutes.

Although I had some ideas about how the activity might develop, students were given freedom to change the direction of the activity. Consequently, I tried to interfere as little as possible and to provide suggestions based only on the students’ observations and conjectures. Of course, no one is perfect. I may have interfered more than I intended at times, and I certainly missed some opportunities to use students’ observations as scaffolding to build up to new ideas. Hopefully other teachers will find ways to improve the activity for their own use. The activity can easily be adapted to fit other objectives and classroom environments.

The measurement activity requires the following materials (see Figure 1):

- Transparent containers of various sizes and shapes (2-liter soda bottles, large juice bottles, etc.): at least one for each group of students;
- Markers: 1 for each group;
- Water: enough to fill each container;
- A ruler and measuring cup for each group.

Because the primary goal for this activity is to allow students to engage in the process of doing mathematics, I focus on the Standards for Mathematical Practice (SMP) (NGA & CCSSO, 2010) and the Process Standards (PS) (NCTM, 2000) demonstrated by students during the lesson. A standard is indicated in parentheses at a moment in the activity at which I believe students exhibited that particular standard. By nature of the group dynamic, students engaged in the Process Standard of *Communication* (PS3) throughout the activity; so, it will not be mentioned further.

I began the lesson with a brief demonstration, filling one of the containers with water. When I asked the students about what measurable quantities came to mind while I was filling the container, students responded with a variety of quantities, including height and volume. I asked them if the water level or volume could be measured while water is being poured into the container. This is not possible (or at least not easy to do accurately) because the water level is constantly changing. If the water level is changing, is there a way to measure how fast (or how much) it is changing? After this initial discussion, students split up into 3 groups, and I distributed a variety of containers of different shapes. I simply asked students if they could measure how “fast” the water level was changing. Students discussed their ideas about what this means, and the open-endedness of the problem left the solution path up to students’ interpretation.

Making Sense of the Problem

Initially the students associated "fast" with time. One group began to measure how much time it took to fill their container. Then a student suggested that the answer would depend on how slow or fast they poured the water. In response I asked what she meant by "how slow or fast they poured", and another student in the group responded that it would mean the amount of water per second. This definition was refined to volume per second. At this point I brought the issue to the attention of the rest of the class. The other two groups had observed the same issue, and all students seemed to reach the consensus that time didn't really matter in this problem. One student explained that she could fill the container halfway, go do something else, and then come back to fill it up the rest of the way. Since each group of students had in some way indicated that the problem of how "fast" the container was filled had more to do with the amount (volume) of water in the container than with the amount of time elapsed (SMP1: *Make sense of problems and persevere in solving them)*, I suggested that they measure how much the water level increases with respect to the volume of water added.

Quantifying Change

Once that issue was resolved, it didn't take long for students to start using phrases like "inches per cup" and associating the notion of how "fast" the water level changes with dividing height by volume. They tried measuring the height of the container and dividing the height by the total volume of the container. This gives an average rate of change for the water level in the container with respect to volume, but it is only one of many possibilities. This could be viewed as the average rate of change of the water level throughout the process of filling the entire container. I asked them to think about whether the water level was changing at the same rate throughout the container. After a brief period of discussion, one group of students pointed out that the water level increased faster at the top of the bottle, attributing this to how much narrower the bottle was near the top (PS2: *Reasoning and Proof).*

I followed up this observation by encouraging them to find a way to verify it mathematically. Several minutes passed with some discussion but no significant progress. So, I suggested that students collect measurements of the water level by successively pouring 1 cup of water into the container. They marked the water levels on the container, which led them to notice that some marks were closer together than others (See Figure 1). While doing this, students decided to make a table to keep track of their measurements (PS5: *Representations).* Making a table of values helped a group of students come to a new realization: the water level increased by different amounts for each successive cup of water added to the container. In doing this, students were able to make a mathematical connection to their intuitive observations about the relationship between the shape of the container and the rate of change of the water level (SMP4: *Model with mathematics).* Students had made a significant step in the direction of verifying their intuition with mathematics.

Figure 1. Containers filled 1 cup at a time.

One group of students took the mathematical connection a step further by using the table of values to sketch a graph of the water level versus the volume of water in the container (PS5). The graph consisted of data points connected by straight lines (a line graph). A student noticed that these line segments were steeper on intervals corresponding to narrower parts of the container. She also observed that this demonstrates that the water level changes at different rates at different parts of the container (PS2). Since a student had made a connection between slope and rate of change, I asked the groups to calculate the slopes. Students compared the values of the rate of change on different intervals within their own graph as well as with those in other groups' graphs. They concluded that the slope of the line segment represented the rate of change in the water level for the corresponding cup of water added (SMP2: *Reason abstractly and quantitatively).*

Building towards Calculus

Once students had developed the connection between rate of change and the slope of the line segment between two points on a graph, I asked them to repeat the process, successively adding ½ cup of water to the container. One group of students finished this much more quickly than others and explained that they already had the data they needed from the first trial, and they only needed to add ½ cup the first time and then successively add 1 cup to get the other values (SMP8: *Look for and express regularity in repeated reasoning).* This time their tables and graphs had twice as many data points. A student in one group pointed out that the rate of change was not constant on any of the intervals. After computing the slope of some of the line segments in the new graph, he claimed that the rate at which the water fills the container had to be different for each additional cup because the slope of each line segment was different. Here, he was making a general statement that the rate was never the same at two points. However, a student in another group refuted this point. She used her graph as evidence, but she explained it further by relating the rate of change back to the shape of the bottle. Her group was using a juice bottle, consisting mostly of uniform square-shaped cross-sections, and she pointed out that the slope was constant whenever the shape of the container was consistent (PS2; SMP3: *Construct viable arguments and critique the reasoning of others).*

To wrap up the discussion, I asked students to consider again the question of how “fast” the water level is changing as we add water to the container. Students agreed that the answer to this question is open to interpretation. They indicated that the rate of change is not necessarily the same throughout the container and that the question can only be answered if more information is provided about “where” we want to know how fast the water level is changing. In response I asked whether we could determine how fast the water level is changing at the instant when there is 1 cup of water in the container. My students said that it is not possible because two points were needed to measure any change. So, I asked if it is possible to approximate the rate of change when there is a specific amount, say 1 cup, of water in the container. The students agreed that they could use the slope of the line obtained after adding an additional cup of water. Eventually, one student asserted that the approximation could be improved by using the slope of the line obtained after adding a ½ cup of water. Another student built on this reasoning to claim that the approximation could be improved further by adding a smaller amount like ¼ cup or less (SMP8). Students were surprised to find out that they had just discovered one of the most important and foundational concepts of calculus (PS4: *Connections).*

Discussion

Measurement activities have the potential to encourage students to take part in the process of doing mathematics as opposed to observing the results of mathematics done by others. My students demonstrated several characteristics of a mathematician at work, from developing definitions to providing counterexamples to generalizing a process. This activity also shows that content and procedural understanding do not have to be sacrificed in order to promote mathematical practice. Students made tables of values, sketched graphs, and calculated rates of change. Even though I was the one who instigated the computation of rates of change, they took the initiative to create tables and graphs. Because the starting point of the activity was for students to determine how to quantify change, it was inevitable that we would address rates of change. In this case, I intervened with guiding questions to ensure that students reached this concept within the classroom period.

Because I asked only guiding questions, students could feel like they were responsible for the ideas that were generated in the classroom. Most importantly, students were able to grapple with complex and unfamiliar concepts in a low-stakes and accessible environment. My students were shocked to find out that the mathematics they were doing was part of the beginnings of calculus. One student said he had never thought calculus would be something that he would ever be able to wrap his head around. But he was pleasantly surprised to find out that doing advanced mathematics was within his grasp. These students were able to build up from the concept of average rate of change to the notion of successive approximation on smaller intervals. What will your students discover?

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