SPRING/SUMMER 2014 | VOLUME 5, ISSUE 1

JOURNAL OF MATHEMATICS EDUCATION AT TEACHERS COLLEGE

A Century of Leadership in Mathematics and Its Teaching

Mathematics Pre-K through 8

Published Semiannually by the Program in Mathematics and Education
TEACHERS COLLEGE | COLUMBIA UNIVERSITY

JOURNAL OF

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Tasks to Advance the Learning of Mathematics

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ABSTRACT Tasks to Advance the Learning of Mathematics (TALMs) were developed to stimulate grades 5 – 8 students' curiosity about complex mathematical relationships, inspire them to reason abstractly and quantitatively, encourage them to consider and create alternative solution approaches, develop their skills to persuade others about the viability of one solution approach over others, and enhance their perseverance toward problem solutions. Tasks are of nine types: Connect Calculation to Context, Rank Order Solutions, Identify What's Wrong If Anything, Defend an Opinion, Work Backwards, Predict and Explain, Think and Choose, Place Them Right, and Make Sense of a Situation. All tasks require application of concepts and skills from one or more domains of mathematics. As students solve these problems, they quickly identify what they know and what they are not sure about; that is, they assess their own degrees of understanding and learn at point of need. The article concludes with recommendations for implementing TALMs and an invitation for students and teachers to create their own.

KEYWORDS *middle school mathematics, mathematical problem solving, challenging mathematical tasks, making sense of mathematics*

Throughout the history of mathematics education in the United States, despite swings in emphasis from attention to the structure of mathematics (e.g., "modern math" or "new math") to a focus on computational algorithms (e.g., the "basics"), there has been one constant: The desire to develop students' "habits of mind." We want students eager to tackle problems for which answers are not immediately known, be curious and inquire, think flexibly, consider the opinions of others, edit their own work, and persevere to the solutions of problems. These habits of mind are elaborated in the first three mathematical practices of the *Common Cores State Standards for Mathematics* (CCSSM).

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments. (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010, p. 6)

How to develop these "habits" became the focus of a major effort of the Practice Research and Innovation in Mathematics Education (PRIME) Center at Arizona State University and the middle school teachers participating in our Center's projects.

Our interest in developing the types of tasks that promote hard thinking was inspired not only by the Mathematical Practices outlined in the CCSSM, but also by two other sources, one published in 1909 and the other, almost 100 years later, in 2006. In this article, these two "inspirations" for Tasks to Advance the Learning of Mathematics (TALMs) are described, along with examples appropriate for exploration by students in grades 5 – 8. The article concludes with discussion of how TALMs can be used to stimulate the learning of new ideas, review and extend what was previous learned, and provide a venue not only for creative thinking but also creative writing!

What Sparked Interest in the Creation of TALMs

In the preface to *Problems Without Figures: For Fourth Grade to Eighth Grade and for Mental Reviews in High Schools and Normal Schools* (1909), Gillan states,

> every problem in arithmetic calls for two distinct and widely different kinds of work: first, the solution, which involves a comprehension of the conditions of the problem and their relation to one another, and second, the operation. First we decide what to do, this requires reasoning. Then we do the work: this is merely mechanical processes. (p. 3)

Of interest in this description is the distinction made between the two types of processes employed in solving problems, comprehending and implementing. Comprehending a problem means understanding its context and question. This requires interpreting the mathematical relationships presented in prose, and in some cases, in combination with other types of representations (e.g., drawings, graphs), and then developing a solution approach that answers the question. The implementing then follows naturally. After the introduction, Gillan presents 360 problems without numbers. The job for the solver is to figure out and then describe how to solve each problem, and in some cases, identify the nature of the data to be used in the problem solution. Five problems from Gillan's (1909) book are presented below, as a teaser to readers! Please note: The problems are reproduced without any changes in punctuation or sentence structure.

> Problem 13: How can I find how many times a wagon wheel will turn in going three miles? (p. 6)
>
> Problem 51: A boy knows at what rate he can walk and at what rate he can ride on a bicycle, also how far it is to town where he left his bicycle for repairs. How can he find how long it will take to walk to town and return on his bicycle? (p. 11)
>
> Problem 58: Knowing the sum of three consecutive numbers, how can you find each of the numbers. (p. 11)
>
> Problem 87: Lay three toothpicks on the desk in the form of an equilateral triangle. Now arrange six to make the same kind of a figure, two on a side. The second triangle is how many times as large as the first? Show this without any ciphering. (p. 15)
>
> Problem 225: The American people spend each year for war much more than for education. If you know the total amount spent for each purpose, how can you find the per capita expense for war and for schools? (p. 32) (C.G. Note: Imagine! Education is still not funded well.)

What is appealing about these 1909 problems is that they require students to bring to bear previous learned concepts and skills, and in many cases, a variety of reasoning methods for their solutions. Furthermore, in the process of solving the problems, students identify what they know and do not know. In so doing, teachers are betters able to gauge students' depths of understanding. In our experience, these types of problems also illicit good questions from students who want to know what mathematics education was like more than 100 years ago!

The other source of our "inspiration" was *E&M TIPERs: Electricity & Magnetism Tasks Inspired by Physics Education Research* (Hieggelke, Maloney, Okuma, & Kanim, 2006), funded by the National Science Foundation. Although focusing on advanced study of the sciences, what is intriguing about this publication is the multiplicity of task types employed. Most TIPER Tasks are "open." That is, they are open to multiple interpretations, multiple solution approaches, and multiple solutions. A subset of these TIPER-type tasks matched well with our goal to design tasks to develop student expertise with the first three Mathematical Practices.

What are TALMs?

Tasks to Advance the Learning of Mathematics (TALMs) are challenging mathematical problems that have the following characteristics. They

- require use of mathematical reasoning and the application of key concepts and skills from the various domains of mathematics (e.g., number, measurement, algebra, geometry) for their solutions;
- develop in-depth understanding of the big ideas of mathematics, its concepts, skills, and reasoning methods;
- enhance interpretation of mathematical relationships presented in text, as well as in other representations;
- cause experimentation with differing solution approaches;
- prompt student discussions, argumentations, and debates about problem interpretations, solution approaches, and solutions;

- promote "stick-to-it-ive-ness;"
- enhance student identification of what is problematic, of what they know, and what they don't know; and
- inform teachers about students' understanding "on the spot" so that adjustments (more challenge or review) in subsequent explorations can be made in a timely manner.

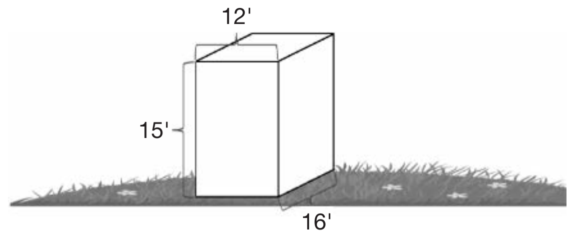
In the discussion that follows, the nine types of TALMs types are described. Each is illustrated with a task designed for students in grades 5 – 8. For many of the TALMs, there are multiple solution methods or solutions. This "multiplicity" provides the fodder for mathematical conversations among students and teachers, and enhances the knowledge and communication talents of students. For all examples, sample solutions are provided.

Connect Calculation to Context

In work with students in grades 5 – 8, we observed that when given problems in which they have to carry out calculations or apply formulas, they give little (or no) thought to whether those actions produce useful information about the contexts. In the Connect Calculation to Context tasks (Figure 1), students are presented with a calculation and text, along with a diagram, graph, or scale model. Their job is to not only determine if the calculation provides useful information about the context, but also to explain why that is the case.

Rank Order Solutions

Rank Order Solutions tasks (Figure 2) present students with sets of problems in the same contexts. Their job is to solve the problems and then rank order the solutions based on a particular condition. Multiple problems in the same context not only enhance student understanding of the mathematics to be applied, but also how different numbers, or pieces of data, interact and influence the solution. In the Rank Order Solutions example in Figure 2, students have to determine the "solution success" (the rate of number of baskets made to total number of baskets attempted) for each player. By recording rates from greatest to least, students gain insight into the importance of the relationship between the two pieces of data. As one student commented, "Just because someone made more baskets doesn't mean that person is the best. You have to think about how many times he tried."



The figure above shows the dimensions of a new memorial located at the entrance of the town park. The memorial is to be painted.

2(12×16) + 2(15×12) = 1224

Does the above calculation give us useful information about this context? Explain.

Solution: No. The calculation does give the surface area (six faces) of the memorial. However, it does not take into consideration that one face, the bottom (12' x 16') is on the ground and cannot be painted. The correct calculation is: **1(12×16) + 2(15×12) = 1032**

Figure 1. Example of a Connect Calculation to Context task.

The table below shows free-throw data for five basketball players.

Name	Baskets Made	Total Number of Shots Taken
Tom	15	30
Ellen	12	25
Hank	2	20
Jeanine	23	25
Ron	32	33

Rank the players from best to worst for free-throws made. Explain your reasoning.

______ ______ ______ ______ ______
Best Worst

Solution: The ratio of baskets made to the total number attempted is computed for each player and then ordered from greatest to least.

Ron: $\frac{32}{33}$ = 0.97; Jeanine: $\frac{23}{25}$ = 0.92; Tom: $\frac{15}{30}$ = 0.5;
Ellen: $\frac{12}{25}$ = 0.48; Hank: $\frac{2}{20}$ = 0.1

Figure 2. Example of a Rank Order Solutions task.

Identify What's Wrong, If Anything

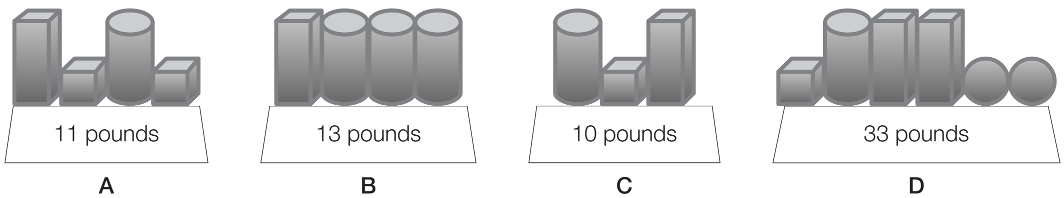
Each Identify What's Wrong, If Anything? task (Figure 3) presents a diagram or situation and a statement about that representation. Students have to decide if something is wrong with the statement or not. If something is wrong, they describe the error and how to correct it so that the statement makes sense in the context. If the statement is correct as it stands, students explain why it is valid. That is, they solve the problem and explain why the answer makes sense contextually.

> **Statement:**
>
> Sarina entered a 3.5 mile race. She ran the first ½ mile in 5 minutes 30 seconds. If Sarina continues running at that speed, she will finish the race in 33 minutes.
>
> What, if anything, is wrong with the statement above? If something is wrong, explain the error and how to fix it. If the statement is correct, explain why it is valid.
>
> ---
>
> **Solution: The statement is correct.** From the second sentence, it can be determined that Sarina runs 1 mile in 11 minutes. Since she has 3 miles left to finish the race, and 3×11= 33 minutes, the statement is valid. Note: The full time to complete the race is not the question. That frequently confuses students who begin to compute without fully reading and interpreting the situation.)

Figure 3. Example of an Identify What's Wrong, If Anything? task.

Defend an Opinion

Defend an Opinion tasks (Figure 4) present students with three or four statements about a mathematical relationship that is depicted in a drawing, diagram, graph, map, or photograph. Some statements may be in agreement and some may not. Students decide which statements are true and justify their decisions, statement by statement. In Figure 4, the picture shows four weight scales with different numbers of 3-D shapes on each along with their total weights. This is a representation of a system of four equations with four unknowns.

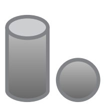
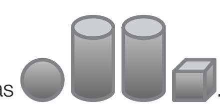


Same shapes have the same weight.

Decide if each statement below is true or false. Explain your answer.

Julie: "  weighs more than  ."

Tom: "  weighs less than  ."

Marcus: "  weighs the same as ."

Solution:

Julie: True. (8 pounds is greater than 1 pound)

Tom: True. (1 pound is less than 2 pounds)

Marcus: False. (7+8, or 15 pounds ≠ 8+2+2+1, or 13 pounds)

Explanation:

1. Since all of C is on A, the cube is 11 – 10, or 1 pound.
2. On C, since the cube is 1 pound, the cylinder and rectangular prism together are 9 pounds.
3. On B, since one cylinder and one rectangular prism together are 9 pounds, the extra two cylinders are13 – 9, or 4 pounds, and one cylinder is 2 pounds.
4. On C, replace the cube with 1 pound and the cylinder with 2 pounds. Then the rectangular prism is 7 pounds.
5. On D, replace the cube with 1 pound, the cylinder with 2 pounds, and each rectangular prism with 7 pounds. Then the 2 spheres are 33 – 17, or 16 pounds, and one sphere is 8 pounds.

Figure 4. **Example of a** Defend an Opinion task.

Work Backwards

Work Backwards tasks (Figure 5) are "Jeopardy-type" problems that present students with "answers" in the form of equations, formulas, inequalities, graphs, or diagrams. For each Work Backwards task, students have to describe the situation (mathematical relationships) that could be "summarized" by the given representation. This task, by its very nature, is open to multiple interpretations and solutions.

Predict and Explain

Predict and Explain tasks (Figure 6) present students with a function. Their job is to predict what will happen to one variable as the value of the other variable changes, either increasing or decreasing in some specified manner. They also describe how they made their decisions. The functions used for these tasks are generally linear or quadratic.

Describe a situation for which this equation would apply. Use a drawing if helpful.

$$S = \frac{\$12.50 + \$10.50 + \$8.75 + \$20.25}{4}$$

Solution: This problem focuses on averages. Answers will vary. Example: Four friends went out for dinner. They shared the check. Each friend's share was $52/4, or $13.

Figure 5. Example of a Work Backwards task.

$$-7 = 3x - y$$

Which of these is true?

As x increases by 1 in the equation above, y increases by:

a. 1
b. 2
c. 3
d. 7
e. 10

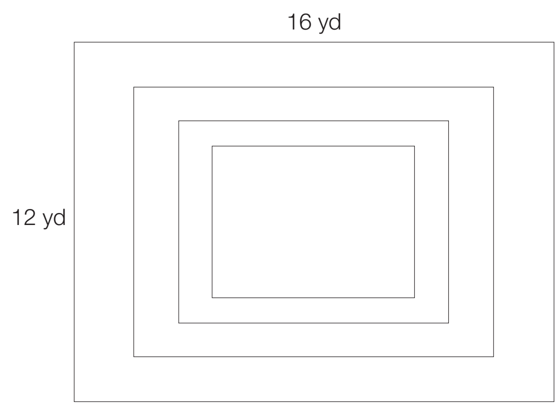
How did you make your decision?

Solution: c. One approach is to rewrite the equation as $y = 3x+7$. Then input values for x, beginning with 1, and keep track of corresponding values of y. As x increases by 1, y increases by 3.

Figure 6. Example of a Predict and Explain task.

Think and Choose

Think and Choose tasks (Figure 7) are multiple-choice problems, each with one or more correct solutions. Students have to select the correct solution(s), describe their reasoning, and then indicate their levels of certainty that their selection(s) is correct. Problem settings use combinations of text with diagrams, graphs, or drawings. The level of certainty indicated is often a good indicator of students' self-assessments of their degrees of understanding.



In this figure, each smaller rectangle has ¾ the area of its surrounding rectangle.

Which of these is true?

The area of the smallest rectangle is about:

a. 81 square yards   b. 108 square yards
c. 144 square yards   d. 192 square yards

Explain your reasoning.

How sure are you of your answer? Darken one of the squares below.

Guessed				Sure			Certain		
1	2	3	4	5	6	7	8	9	10

Solution: a. is correct.

The outer rectangle's area is 12×16, or 192 square yards.

The next smaller rectangle is $\frac{3}{4}\times192$, or 144 square yards.

The next smaller rectangle is $\frac{3}{4}\times144$, or 108 square yards.

The smallest rectangle is $\frac{3}{4}\times108$, or 81 square yards.

Figure 7. Example of a Think and Choose task.

Use all numbers in the rectangle. Fill in the blanks so that the story makes sense.

The Stay Warm Company had an end-of season sale. The regular price of a sweater is $ ______. The regular price of a sweatshirt is less than half the price of a sweater, or $ ______. Both items are on sale. The sweater is on sale for ______% off, and is being sold for $ ______. The sweatshirt is on sale for $ ______. The total cost of the two sale items is $ ______.

21.00	30	42.00
66.50	87.50	95.00

Solution: The Stay Warm Company had an end-of season sale. The regular price of a sweater is $ 95.00.The regular price of a sweatshirt is less than half the price of a sweater, or $ 42.00. Both items are on sale. The sweater is on sale for 30% off, and is being sold for $ 66.50. The sweatshirt is on sale for $ 21.00. The total cost of the two sale items is $ 87.50.

Figure 8. Example of a Place Them Right task.

Fill in numbers so that the story makes sense.

On the interstate, Mr. Ivy drove ______ miles in ______ minutes. At that speed, he made the 140-mile trip from Phoenix to Flagstaff in ______ hours.

On the same interstate, Ms. Fern's average speed of ______ miles per hour was less than Mr. Ivy's. Ms. Fern made the 140-mile trip in ______ hours, or ______ minutes more than it took Mr. Ivy.

Solution: Answers will vary. To solve the problem, students have to recognize that they will be computing two different rates that are related such that Ms. Fern is driving slower than Mr. Ivy.

Example:

On the interstate, Mr. Ivy drove 20 miles in 20 minutes. At that speed, he made the 140-mile trip from Phoenix to Flagstaff in 2 ⅓ hours.

On the same interstate, Ms. Fern's average speed of 5 miles per hour was less than Mr. Ivy's. Ms. Fern made the 140-mile trip in 2 ½ hours, or 10 minutes more than it took Mr. Ivy.

Figure 9. Example of a Make Sense of a Situation task.

Place Them Right

Place Them Right tasks (Figure 8) present students with paragraph descriptions of situations with numbers removed from the text and replaced with "blanks." Removed numbers are listed in a rectangle. The job for students is to interpret the mathematical relationships presented in the text, identify numbers in the rectangle that would fit those relationships, and use those numbers to fill in the blanks so that the situation makes sense.

Make Sense of a Situation

Make Sense of a Situation tasks (Figure 9) are the "sisters" of Place Them Right. Make Sense of a Situation differs in that students are presented with descriptions of situations in which all numeric data have been removed from the descriptions but no replacement numbers are provided. The job for students is to interpret the mathematical relationships presented in the paragraphs, figure out and then fill in the numbers so that the situation makes sense.

Discussion

While working with groups of fifth through eighth graders, we discovered that these problems make students academically "uncomfortable." Some students pointed out that they had never seen problems that "didn't tell me what to do," "have more than one answer, so I don't know which one is right," and "make me tell if I think I am right" ("How sure are you of your answer?"). Why did we get these comments?

By the time students have reached grade 5, they have come to expect teachers to present the "what to do" and know that their job is to replicate the teacher's actions. TALMs reverse that process. TALMs require students to bring to bear what they already know, apply those known concepts and skills from more than one domain of mathematics (e.g. algebra, measurement), interpret a variety of representations to solve the problems, and learn new concepts and skills at point of need. Our motto is "struggling is good, suffering is not!" When students get stuck, we urge them to articulate what they need, do not know, or feel unsure about. In this way, students are "assessing" their own understanding. When that occurs, the teacher may provide sources for information or seek other students who may offer advice (but no answers!).

When using these types of problems with students, we allowed them to choose partners as collaborators, gave them access to information sources in the event that definitions were needed, and required them to secure agreement about solutions from at least one other pair of students. Because solution methods differed more often than not, and in many cases, the solutions differed, sufficient time was provided for students to wrestle (mathematically!) with the differences and to present their final work to others in the class. Typically, 1 ½ to 2 hours were allotted to TALMs work. We sent some TALMs home (no more than one or two problems at a time) with students to do with families as a Home Work-Outs adventure. As well as school time devoted to TALMs, some Saturday morning programs were created for students to solve and create TALMs.

Conclusion

In summary, it is the case that by solving TALMs, our middle school students are making sense of problems and persevering through to their solutions (Mathematical Practice 1), reasoning abstractly and quantitatively (Mathematical Practice 2), and constructing viable arguments and critiquing the reasoning of others (Mathematical Practice 3). They are becoming comfortable with the language of mathematics—of mathematical relationships presented in prose, drawings, tables, graphs, and symbols. And, they are developing their creative talents, not only in terms of their solution approaches, but also in the design of TALMs for others.

Grades 5—8 teachers in our STEM in the Middle Project, funded by the Helios Education Foundation, are now developing problems like these to use with their students during school time, for homework, and in after-school programs. By developing TALMs, teachers are not only gaining greater insight into strategies for developing students' talents in mathematical problem solving, but also they themselves are getting smarter mathematically.

We invite others, students and teachers, to develop TALMs and contribute them to our growing collection of challenges. We will make these available on line to all. Contributions can be submitted to cgreenes@asu.edu

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