SPRING/SUMMER 2014 | VOLUME 5, ISSUE 1

JOURNAL OF MATHEMATICS EDUCATION AT TEACHERS COLLEGE

A Century of Leadership in Mathematics and Its Teaching

Mathematics Pre-K through 8

Published Semiannually by the Program in Mathematics and Education
TEACHERS COLLEGE | COLUMBIA UNIVERSITY

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Where is the Treasure? Ask Interactive Geometry Software!

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ABSTRACT In this paper I describe how I have used the classic buried treasure problem with prospective and practicing mathematics teachers to enhance their problem solving abilities and disposition to integrate interactive geometry software (IGS) into the learning environment. I illustrate how IGS may be used as a strategic tool to gain insight into the solution of a problem. When appropriate, students are guided in the development of arguments and proofs to justify their conjectures.

KEYWORDS *treasure problem, interactive geometry software, educational technology, mathematical modeling, problem solving, proof, visualization*

Traditional mathematics instruction focuses on stating and practicing rules, procedures, formulas, or algorithms following the "tell, show, and do" model (Albert & Kim, 2013). Curriculum reforms in school mathematics (e.g., National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA Center & CCSSO], 2010; National Council of Teachers of Mathematics [NCTM], 1989, 1991, 2000; National Research Council, 2001) call for radical changes not only of content but also of pedagogical practices.

Two practices advocated by reform documents are problem solving and the use of technology in teaching and learning mathematics. Problem solving involves undertaking a task or problem situation whose solution method is not known beforehand (NCTM, 2000). Problem solving should be not only a primary goal of learning mathematics, but also a means to do so. Yet, enhancing students' abilities and inclination to engage in problem solving tasks has not been achieved to an acceptable level (Lesh & Zawojewski, 2007; Lester, 1994). In fact, according to Lester (1994), the state of students' performance in problem solving is desperate.

A strategic tool that learners now have at their disposal for solving problems is interactive geometry software (IGS) such as GeoGebra (Hohenwarter, 2002), The Geometer's Sketchpad (Jackiw, 2001), and Cabri (Laborde & Bellemain, 2005). One of the most powerful features of IGS is its capacity to allow learners to visualize solutions to problems that otherwise would be difficult to find. This feature of IGS is a major facilitator of discovery-based learning. Within interactive geometry environments, students enhance not only their geometric intuition, but also their ability to generate conjectures by transforming their geometric configurations and observing their effects on invariant relationships.

In this paper I illustrate how IGS can facilitate problem solving using GeoGebra, although any type of IGS can achieve the same goal. As learners discover a potential solution to a problem, they formulate and test conjectures, anticipating the need for formal mathematical arguments to justify their solutions. In this way, learners are motivated to explain why the proposed conjecture is indeed the solution to the problem (de Villiers, 1998; de Villiers & Mudaly, n.d.; Hadas, Hershkowitz, &

Schwarz, 2000; Jones, 2000; Mariotti, 2000, 2001; Sanchez & Sacristan, 2003). I demonstrate the power of IGS to help learners solve problems with GeoGebra, using as an example a wonderful treasure-hunt-based problem that my students have enjoyed solving with IGS. I have used this problem in both regular classroom instruction and during professional development courses with both middle school and secondary mathematics teachers. The problem is described in Gamow (1947).

Gamow's Buried Treasure Problem

A version of this problem follows:

> An adventurous young man found a map showing the location of buried treasure on a remote deserted island with the following directions:
>
> *On the island there are an oak tree, a pine tree, and an old gallows. Start at the gallows and walk to the oak tree, counting the steps. At the oak tree, turn right 90° and take the same number of steps and put a spike in the ground. Return to the gallows and walk to the pine tree counting the steps. At the pine tree take a 90° turn to the left and walk the same number of steps. Place a second spike at this point. Behold! The treasure is halfway between the two spikes.*
>
> The young man and his friends embarked on an expedition to the island where they found the trees, but, to their dismay, the gallows had disappeared without leaving any trace. Desperate, the men dug randomly without any luck. They returned home heartbroken without the treasure. The task is to devise a plan, if possible, to find the treasure without knowing the original location of the gallows.

I have given this problem to my students as homework. It is interesting to note that students do not automatically think of using IGS to generate possible solutions to the problem. In any case, my students have yet to solve the problem without IGS. However, once I suggest using IGS, the solution becomes apparent. Figure 1 displays the model of the situation using GeoGebra. The points representing the gallows, the oak tree, and the pine tree are draggable, whereas the points representing the spikes and the treasure are not. The gallows and the trees are movable because their positions are fixed but *independent* of each other. The spikes are not draggable because their positions *depend* on the positions of the gallows and the trees: the positions of the gallows and the trees determine the position of the spikes. The treasure is not movable because its position *depends* on the position of the trees. Surprisingly, as we see in the following section, the position of the treasure is independent of the position of the gallows.

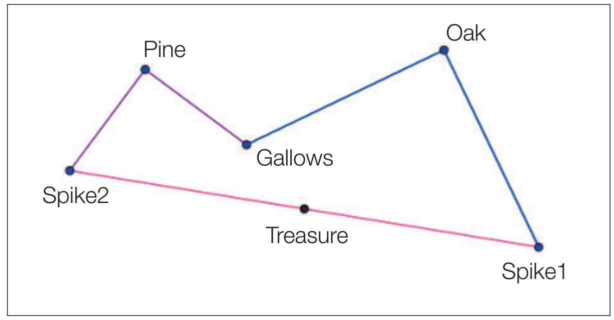

Figure 1. Model of the treasure problem.

Some students argue that we cannot represent the gallows because it has been destroyed while others contend that we should draw it to investigate whether a pattern appears when we "move it around."

A Delightful Solution to the Treasure Problem

After students construct the configuration displayed in Figure 1, I ask them to use the GeoGebra model to make a conjecture about the possible location of the treasure. Some students are puzzled as to what to do, while others start dragging the gallows. Most of the students who drag the gallows are delightfully surprised when they discover that the location of the treasure is independent of the position of the gallows. As they drag the gallows to any position on the working space they see that the location of the treasure remains invariant (Figure 2). All students who follow this approach confirm that the conjecture (that the location of the

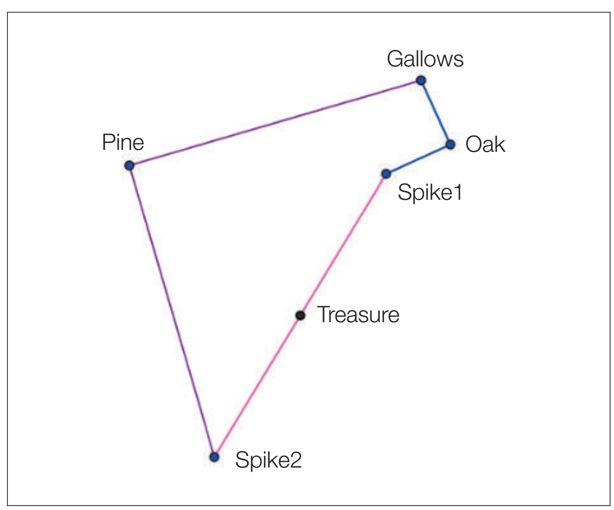

Figure 2. The position of the treasure is invariant with respect to the location of the gallows.

treasure is independent of the location of the gallows) seems plausible.

Some students are intrigued as to why the position of the treasure is invariant with respect to the location of the gallows. At this point, I guide the class to construct a coordinate proof along the following lines:

To simplify the algebra, we can place the oak at the origin, point B (0, 0), and the pine at point C (a, 0), as displayed in Figure 3. The location of the gallows is unknown, say (x, y). Using the fact that $\Delta ABG \cong \Delta BDH$ and $\Delta ACG \cong \Delta CEI$, we infer that the coordinates of D and E are ($-y$, x) and ($a + y$, $a - x$), respectively. Applying the midpoint formula to segment $\overline{DE}$, we deduce that the coordinates of the location of the treasure are ($a/2$, $a/2$), which proves that the location of the treasure is invariant with respect to the position of the gallows.

My students have often been pleasantly surprised at the solution of the treasure problem.

Another Solution Discovered by Students

Other students have approached the problem in a different way. They start by constructing the perpendicular bisector of the segment that connects the two trees (Figure 4). Then they construct the circle with center A, the midpoint of the segment connecting the two trees, and radius $\overline{A0}$ or $\overline{AP}$, the distance from said midpoint to any of the trees. Finally, they notice that the treasure is located at the point of intersection of the perpendicular bisector and the circle. They confirm their conjecture by dragging the oak, pine, and gallows to different positions in the working area of the software and noticing that the position of the treasure is always the point of intersection.

Our next task is to explain why the treasure is located at point T, the point of intersection of $\overleftrightarrow{AT}$ and circle (A, $\overline{A0}$). Students are usually unable to explain why this procedure works. An argument provided by one student follows:

Since the position of the treasure is independent of the gallows, we can place the gallows at A, the midpoint of segment $\overline{P0}$ (Figure 5). Quadrilateral POS_1S_2 is a rectangle (because it is a parallelogram with two right angles), which leads us to conclude that T, the midpoint of segment $\overline{S_1S_2}$, is also the point of intersection of $\overleftrightarrow{AT}$ and circle (A, $\overline{A0}$).

Of course, most students realize that this argument is based on the assumption that the position of the treasure is independent of the location of the gallows. The next task is to connect the students' solution to the algebraic solution via a geometric method.

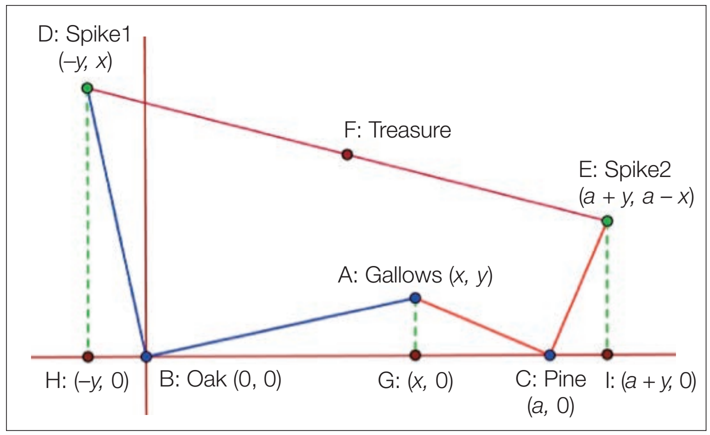

Figure 3. Coordinates for the treasure problem.

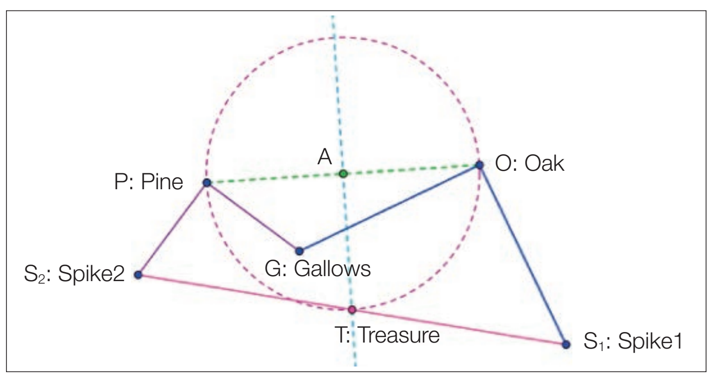

Figure 4. A solution discovered by students.

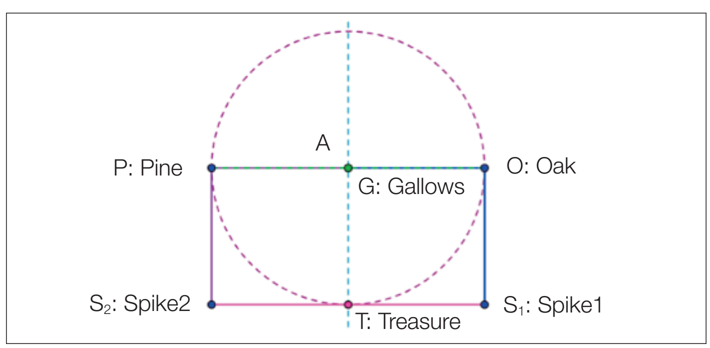

Figure 5. The gallows is located at A, the midpoint of $\overline{P0}$.

Connecting the Two Solutions

The first time that a student discovered the second way to solve the treasure problem, I was intrigued by the simplicity of the method. I also wanted to see why the student's solution worked. The following purely geometric argument connects the two solutions.

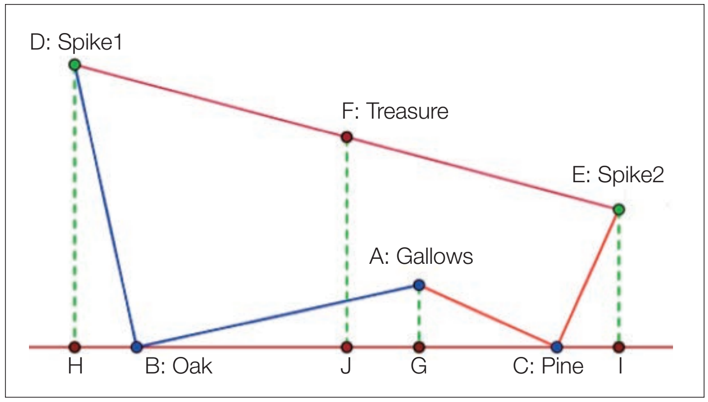

Figure 6. Diagram for the geometric proof.

Using the same figure that we constructed for the algebraic proof, construct segment $\overline{FJ}$ perpendicular to line $\overleftrightarrow{BC}$ (Figure 6). Because F is the midpoint of segment $\overline{DE}$ and segments $\overline{DH}$, $\overline{EI}$, and $\overline{FJ}$ are parallel (each segment is perpendicular to line $\overleftrightarrow{BC}$), we conclude that J is the midpoint of segment $\overline{HI}$, which implies that $\overline{FJ}$ is the midsegment of trapezoid $DEIH$ and, hence, $FJ = \frac{DH + EI}{2} = \frac{BG + GC}{2} = \frac{BC}{2}$. Further, $HB = GA = CI$ so J is the midpoint of $\overline{BC}$ and $BJ = \frac{BC}{2}$. Thus, $FJ = \frac{BC}{2} = BJ$. In other words, the position of point F (the treasure) is independent of the position of point A (the gallows).

Now that we know that the location of the treasure is invariant with respect to the location of the gallows, we can locate the treasure following these directions: construct the perpendicular bisector of $\overline{BC}$ and mark off along this perpendicular line a distance equal to BJ. Most of my students and I are delighted to see the elegant connection between the algebraic and geometric proofs because it allows us to locate the treasure using a simple construction, first "discovered" by one of my students.

Since the adventurous young man most likely did not know mathematics, and certainly did not have IGS at his disposal, could he still have found the treasure? Yes! He could have asked "what if" questions about the location of the treasure: "If the gallows were here, then the treasure would be over there, and if …" He may then have noticed that the location of the treasure did not depend on the position of the gallows. But, as it turned out, he did not have even a clue about the power of looking for a pattern.

Students' Development of Mathematical Practices

I often challenge my prospective middle school and secondary mathematics teachers with the buried treasure problem because it provides us with opportunities to further develop the mathematical abilities recommended by NCTM (1989, 2000) and the *Common Core State Standards for Mathematics* (NGA Center & CCSSO, 2010): solving problems, reasoning abstractly, constructing mathematical arguments including proofs, modeling real-world situations with mathematics, using appropriate tools strategically, and attending to precision.

Using IGS to solve problems such as the treasure hunt offers students a number of excellent opportunities to enhance their problem solving skills:

- It offers students the opportunity to make sense of a problem by analyzing the problem and explaining to themselves and their peers the meaning of the known and unknown information as well as the constraints. To start solving a problem with mathematics and IGS, students need to understand the structure of the problem.
- It provides students the experience of using mathematics as a modeling tool. They need to translate the verbal description of the problem into a dynamic diagram. The construction of the dynamic diagram gives students opportunities to use or apply their mathematical and IGS knowledge.
- It engages students in using IGS to gain insight into a solution not otherwise apparent. IGS allows students to visualize the results of varying the position of the gallows and make conjectures or predictions. IGS also allows students to attend to precision by constructing accurate dynamic representations of the treasure problem.
- It offers students opportunities to reason abstractly and construct viable arguments and invites them to formulate and justify conjectures as well as understand stated assumptions.
- By solving the treasure problem visually, algebraically, and geometrically, students make sense of the interrelationships between the quantities and

variables embedded in the problem statement. They also identify correspondences between the algebraic and the geometric proof.

Concluding Remarks

The treasure problem is an excellent example of how IGS may be used to discover the solution of a problem. Two of the IGS features that fostered insights into the process of discovering solutions to the treasure problem are constructing and dragging. First, the construction capabilities of IGS allowed all students to construct quickly and easily a precise model of the treasure problem. Doing the construction with paper and pencil would have been more time consuming and likely would have produced an imprecise diagram. Second, the dragging feature of IGS allowed some students, after the construction of the model, to rapidly move certain flexible points (i.e., the gallows in Figure 1) and examine how other objects (i.e., Spike1, Spike2, and the treasure in Figure 1) responded dynamically to the changes. The combination of constructing and dragging helped these students to quickly discover that the position of the treasure is independent of the location of the gallows. In addition, both the construction and the dragging capabilities of IGS have allowed other students to solve the problem in another way, as illustrated in Figure 4. In both cases, these two features allowed students to test their conjectures with a range of examples with greater ease and precision than possible with time consuming paper-and-pencil constructions. The construction and dragging capabilities of IGS have also led other students to rapidly refute conjectures. For example, a few students have conjectured that the location of the treasure is the intersection of the angle bisectors of the Spike1-Oak-Gallows angle and the Spike2-Pine-Gallows angle (Figure 7a). After they dragged the gallows to a new position, they immediately realized that their conjecture was not correct (Figure 7b).

Solving problems is an essential activity of doing mathematics (Halmos, 1980; NCTM, 1989, 2000; Pólya, 1945). Not surprisingly, past and current reforms (e.g., NCTM, 1989, 1991, 2000; NGA Center & CCSSO, 2010; National Research Council, 2001) call for teachers to make problem solving an integral part of learning mathematics. As argued by Pólya (1945), helping students to solve problems is one of the most important roles of the mathematics teacher. However, problem solving is a complex process because, by its very definition, it is not a step-by-step procedure. Students should be encouraged to use any available resource to solve problems, including interactive geometry software such as GeoGebra (Hohenwarter, 2002), The Geometer's Sketchpad (Jackiw, 2001), and Cabri (Laborde & Bellemain, 2005).

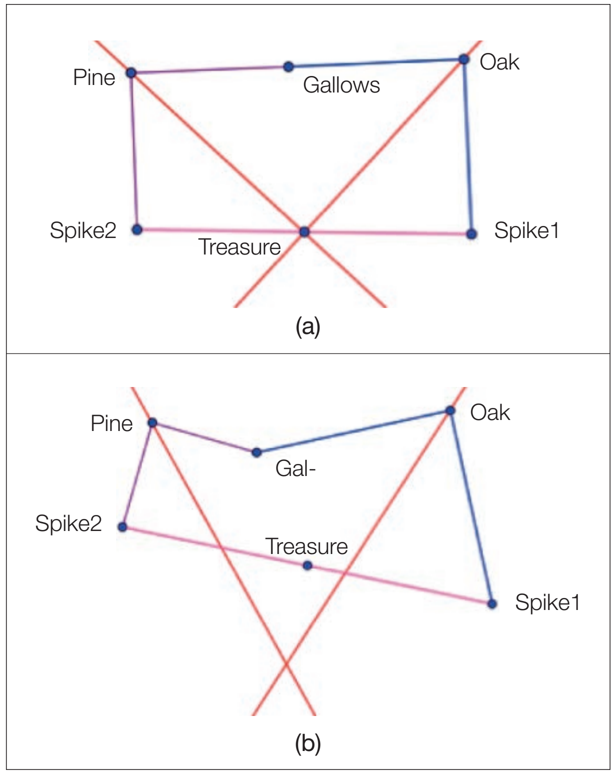

Figure 7. An incorrect solution to the treasure problem.

At the start, some of my students have a hard time believing that it is completely unnecessary to know the position of the gallows to locate the buried treasure. I almost hear them saying, "I see it, but I do not believe it." Consequently, many of them were motivated to develop a proof to understand why that is the case. In addition, as discussed previously, this problem offers ample opportunities for students to further develop their abilities to solve problems, reason abstractly, construct mathematical arguments including proofs, model real-world situations with mathematics, use appropriate tools strategically, and attend to precision, as recommended by the *Common Core State Standards for Mathematics* (NGA Center & CCSSO, 2010).

Another aspect of problem solving is to pose related problems, such as asking "what-if" questions—the answers to which may provide insight to the problem at hand (Brown & Walter, 1990). For the treasure hunt problem, for example, we can ask, "What if we make a turn different from 90°?" The use of IGS facilitates the investigation of these new problems and allows us to "experience the tension and enjoy the triumph of discovery" (Pólya, 1945, p. v).

Finally, it is important that both prospective and practicing mathematics teachers experience solving problems with interactive geometry software so that they develop a disposition to create inquiry-based learning environments where technology facilitates finding the solution to problems with an appropriate level of difficulty. It is my belief that the treasure problem is one such problem. It is also a delight to discover its surprising solution with IGS.

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