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Solving Optimization Problems with Dynamic Geometry Software: The Airport Problem

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ABSTRACT This paper describes how the author's students (in-service and pre-service secondary mathematics teachers) enrolled in college geometry courses use the Geometers' Sketchpad (GSP) to gain insight to formulate, confirm, test, and refine conjectures to solve the classical airport problem for triangles. The students are then provided with strategic hints to develop a mathematical proof to justify the conjecture discovered with the help of GSP.

KEYWORDS *Airport problem, Dynamic Geometry Software, Geometers' Sketchpad, problem solving, optimization problems, proof, secondary mathematics teachers, visualization*

Introduction

Learning to solve problems that arise within mathematics and real-world contexts has been one of the goals of past and recent reform efforts in mathematics education (Borromeo Ferri, 2013; CCSSI, 2010; National Council of Teachers of Mathematics [NCTM], 1989, 2000; Schoenfeld, 2013). To solve a problem, the learner should use any available resources, which include not only mathematical knowledge but also technological tools.

The use of technological tools is more than a simple new way to perform procedures and algorithms. They offer learners unparalleled opportunities to explore a problem by formulating and testing conjectures, both visually and empirically. They allow us to explore and experiment the content of a problem by varying its parameters dynamically and noticing the effects of these changes on relationships. Technological tools also allow learners to extend the range of problems accessible to them. Students can represent and model complex problems that would be cumbersome, if not impossible, to investigate without the use of technological tools. Ac-

cording to *Principles and Standards for School Mathematics* (NCTM, 2000), "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (p. 24).

One of the most powerful technological tools that has been developed to support the process of representing and modeling mathematical and real-world problems is Dynamic Geometry Software (DGS). DGS, like other technological tools, can graph, visualize, and compute efficiently and accurately. The most common types of DGS include The Geometer Sketchpad (Jackiw, 2001), Cabri Geometry (Laborde & Bellemain, 2005), and GeoGebra (Hohenwarter, 2002), to name just a few.

Laborde and Laborde (1995) describe three types of student behavior that are encouraged and facilitated within DGS environments. First, DGS facilitates the formulation and testing of a conjecture by dragging a diagram around the screen and considering extreme cases. Second, DGS allows students to openly experiment with a diagram and to ask "what if" questions. Third, DGS allows learners to systematically repeat experiments to further test the validity of a conjecture

across a wide range of conditions or factors to determine the domain of its validity.

Geometric optimization problems, in general, are suited to be investigated with DGS because the software allows us to build a geometric model of the problem situation and manipulate some of its parts. The software instantly changes other parts and performs precise calculations while preserving the intended constraints. We can then use the new information to further drag some parts of the model to build, test, and refine conjectures. In this paper, I illustrate how GSP can be used to model optimization problems, although any type of DGS can be used. In doing so, I illustrate how the graphical power of DGS "affords access to visual models that are powerful but that many students are unable or unwilling to generate independently" (NCTM, 2000, p. 23) using as an example the classic airport problem. This problem also serves as an excellent vehicle to exemplify how DGS can foster our intuition and provide insight into the solution to a mathematical or real-world problem.

The Airport Problem

A version of the airport problem follows:

Three towns — Armon (A), Betania (B), and Calista (C) — are planning to build an airport to serve the three cities. To keep costs at a minimum, the airport needs to be constructed at a place where the sum of its distances to each of the cities is minimal. (a) Describe the minimum distance point for the location of the airport; (b) construct the optimal point.

I use this problem in my college geometry classes for both pre-service and in-service secondary mathematics teachers. Often, students' initial approach to solving this problem involves using only paper and pencil. Because this is not a routine problem, they struggle solving it because they have not seen a "problem like this in class." In spite of "not having a clue" about how to solve it, students do not automatically use DGS to represent and model it, even though we have used DGS to perform other types of investigations. At this stage of the problem-solving process, I suggest representing and modeling the problem with GSP. Before they represent the problem with GSP, however, I ask them to make a prediction of where the airport should be constructed. The most common responses are either at the circumcenter (the point of intersection of the perpendicular bisectors of the sides of the triangle) or at the incenter (the point of

intersection of the angle bisectors of the interior angles of the triangle). Some students who predict that the circumcenter is the optimal location for the airport argue that the circumcenter is equidistant from the three towns and, thus, this point minimizes the sum of the distances to the three towns. On the other hand, some students who claim that the airport should be built at the incenter claim that this triangle center is the optimal point because it is always in the inside of the triangle and it is equidistant from its sides.

A Surprising Solution to the Airport Problem

Our second task is to use GSP to represent and model the problem. Figure 1 represents two common diagrams designed by students.
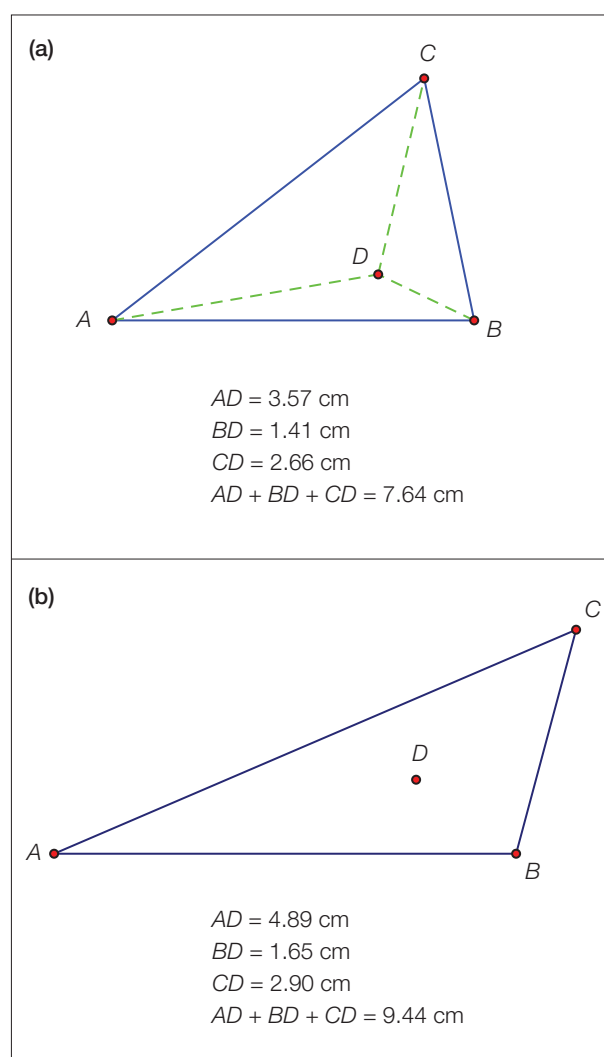

(a)

AD = 3.57 cm
BD = 1.41 cm
CD = 2.66 cm
AD + BD + CD = 7.64 cm

(b)

AD = 4.89 cm
BD = 1.65 cm
CD = 2.90 cm
AD + BD + CD = 9.44 cm

Figure 1. Two initial representations of the airport problem.

After having represented the problem with GSP, students drag one of the vertices of the triangle until they notice that the sum of the three distances from point D to the three vertices of the triangle seem to be as small as possible (Figure 2).
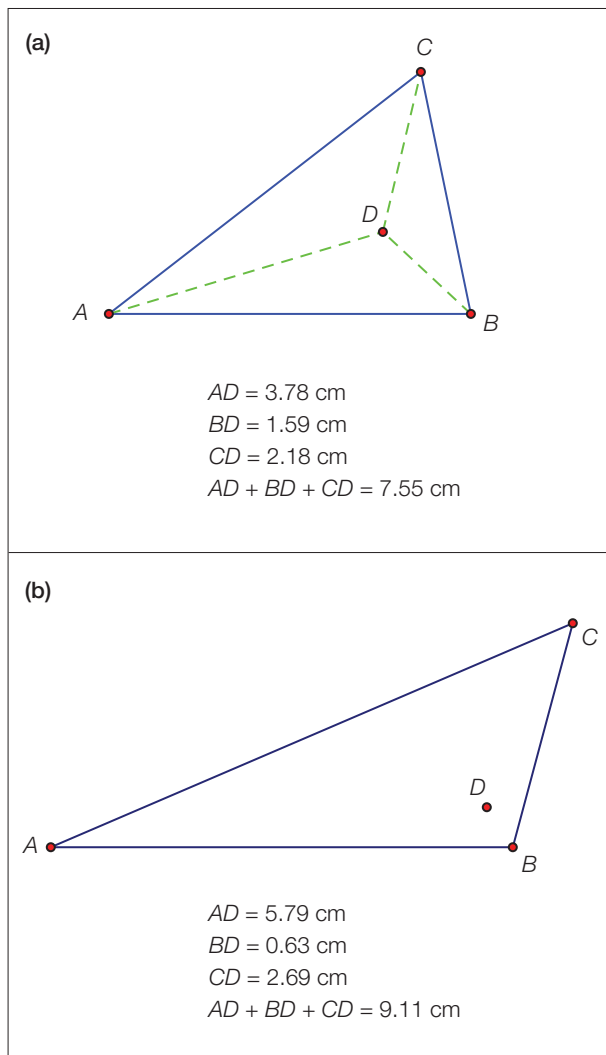

(a)

$AD = 3.78$ cm
$BD = 1.59$ cm
$CD = 2.18$ cm
$AD + BD + CD = 7.55$ cm

(b)

$AD = 5.79$ cm
$BD = 0.63$ cm
$CD = 2.69$ cm
$AD + BD + CD = 9.11$ cm

Figure 2. Searching for the optimal location for the airport.

Our third task is to examine our model constructed with GSP and characterize point D. By analyzing visually the dynamic representation of the problem, most students immediately notice that point D is neither the circumcenter nor the incenter of the triangle, as some of them initially conjectured. Most students who represent the problem using a diagram like the one depicted in Figure 1a (or 2a) are able to describe the distinctive nature of point D by measuring $\angle$ADB, $\angle$BDC, and $\angle$ADC: point D is the point at which the three

non-overlapping angles determined by the point and the segments joining said point with the vertices of the triangle are congruent, each measuring 120° (Figure 3). In other words, the point at which the sides of the triangle subtend congruent angles. In contrast, few students, if any, who represented the problem using a diagram like the one displayed in Figure 1b (or 2b) are able to characterize point D. After sharing the findings with the rest of the class, some students say how neat and surprising the solution to the airport problem is. I mention to the class that this point is called the equiangular point of the triangle. The class often formulates the initial conjecture as follows:

Conjecture 1. The equiangular point of the triangle minimizes the sum of the distances from it to each of the three vertices of the triangle.

$AD = 3.78$ cm
$BD = 1.59$ cm
$CD = 2.18$ cm
$AD + BD + CD = 7.55$ cm

$m\angle ADB = 120.36°$
$m\angle BDC = 119.62°$
$m\angle ADC = 120.02°$
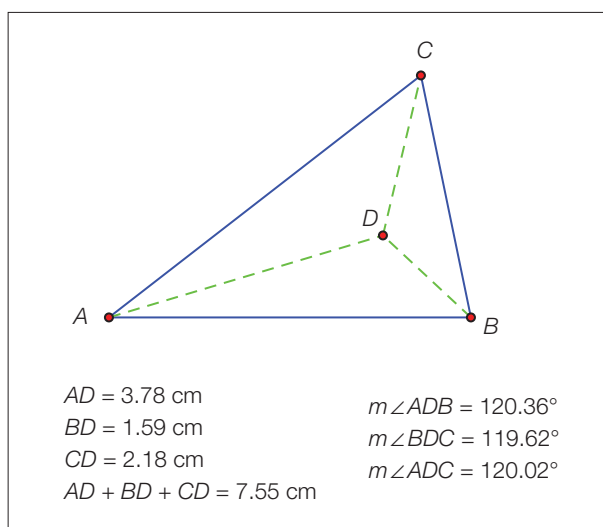
Figure 3. The minimum distance point for a triangle seems to be the equiangular point.

Constructing the Equiangular Point

After having conjectured that the minimum distance point of a triangle is the point at which the sides of the triangle subtend congruent angles (i.e., angles measuring 120° each), I ask students what we can do to have point D remain the equiangular point as we drag any of the vertices of the triangle to a new location. By now students know the difference between drawing and constructing and thus they respond that we need to construct the equiangular point. Some add that constructing the equiangular point would allow us to obtain

more exact measurements for the three congruent angles surrounding the equiangular point. The problem, however, is how to construct the equiangular point.

Because constructing the equiangular point is a challenging construction, students are allowed to search the internet or use other resources to find out how to perform this construction. After searching for minimum total distance or equiangular point for triangles, students often find two methods to construct the elusive point. Both methods involve constructing outward equilateral triangles on the sides of the given triangle. The two ways to construct the equiangular point are straightforward within DGS environments.

One method constructs the equiangular point by constructing the segments joining each vertex of the given triangle with the remote vertex of the equilateral triangle constructed externally on the opposite side. The three segments are concurrent at the equiangular point of the given triangle (Figure 4).

The second method to construct the equiangular point is to construct the circumcircles of the three outward triangles. The three circumcircles are concurrent at point G, the equiangular point of the given triangle (Figure 5).
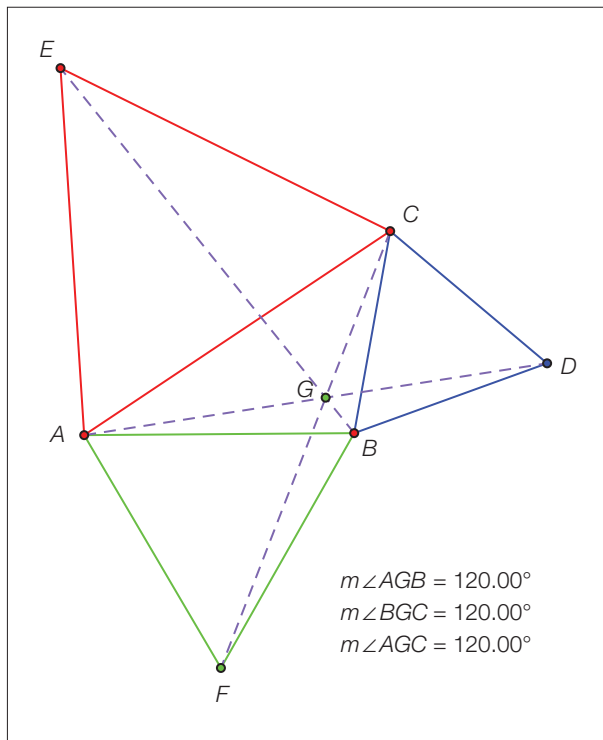

$m\angle AGB = 120.00°$
$m\angle BGC = 120.00°$
$m\angle AGC = 120.00°$

Figure 5. The three circumcircles of triangles ΔBCD, ΔACE, and ΔABF are concurrent at the equiangular point.

$m\angle AGB = 120.00°$
$m\angle BGC = 120.00°$
$m\angle AGC = 120.00°$

Figure 4. Segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent at point G, the equiangular point of ΔABC.
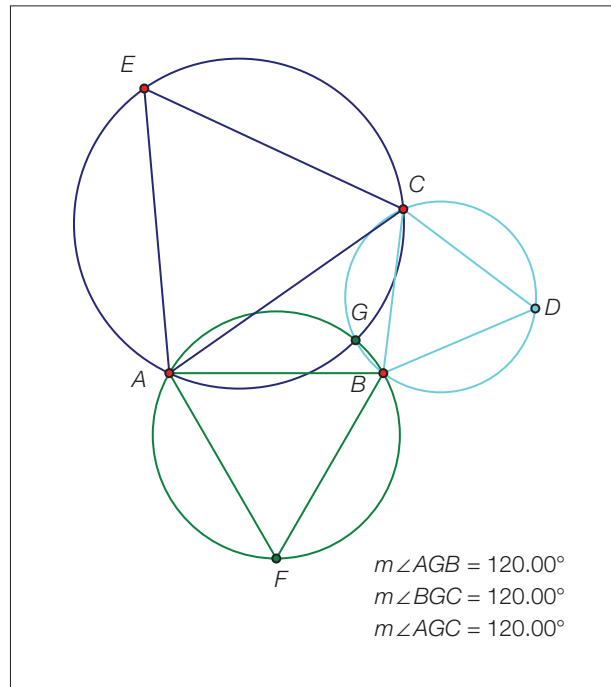
Confirming and Refining the Conjecture

After I guide the class to prove that the two methods to construct the equiangular point are correct (for certain triangles), we come back to test further the conjecture that said point is also the minimum distance point for other types of triangles. Some students test their conjecture with the equiangular point constructed using method 1 (Figure 6a) while others test it with the equiangular point constructed using the second method (Figure 6b).

As students drag point H to different locations to try to find another point with a shorter total distance, they further confirm that G, the equiangular point, is the minimum distance point. At this stage of the investigation students are ready to verify their conjecture for additional triangles. As students consider several types of triangles, some quickly hypothesize that the minimum distance point does not exist for some triangles because point G disappears (Figure 7a), while other realize that the optimal point ceases to be the equiangular point (Figure 7b). All students later conjecture that the minimum distance point does exist for all triangles, but it is not always the equiangular point.
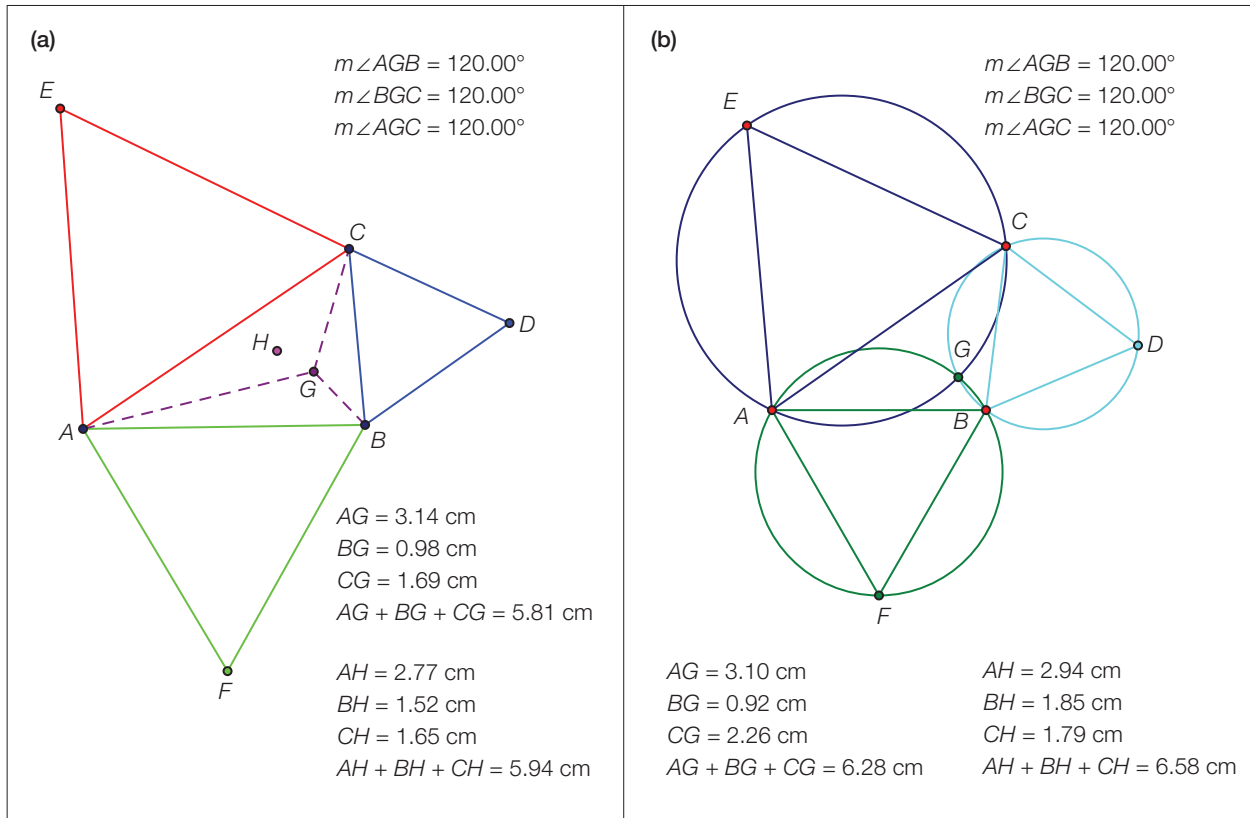
(a)

$m\angle AGB = 120.00°$
$m\angle BGC = 120.00°$
$m\angle AGC = 120.00°$

$AG = 3.14$ cm
$BG = 0.98$ cm
$CG = 1.69$ cm
$AG + BG + CG = 5.81$ cm

$AH = 2.77$ cm
$BH = 1.52$ cm
$CH = 1.65$ cm
$AH + BH + CH = 5.94$ cm

(b)

$m\angle AGB = 120.00°$
$m\angle BGC = 120.00°$
$m\angle AGC = 120.00°$

$AG = 3.10$ cm
$BG = 0.92$ cm
$CG = 2.26$ cm
$AG + BG + CG = 6.28$ cm

$AH = 2.94$ cm
$BH = 1.85$ cm
$CH = 1.79$ cm
$AH + BH + CH = 6.58$ cm

Figure 6. Further empirical evidence that the equiangular point is the minimum distance point.

(a)

$AH = 3.02$ cm
$BH = 0.79$ cm
$CH = 2.53$ cm
$AH + BH + CH = 6.34$ cm

(b)

$m\angle AGB = 60.00°$
$m\angle BGC = 60.00°$
$m\angle AGC = 120.00°$

$AG = 4.43$ cm
$BG = 0.69$ cm
$CG = 2.54$ cm
$AG + BG + CG = 7.66$ cm

$AH = 3.69$ cm
$BH = 0.55$ cm
$CH = 2.50$ cm
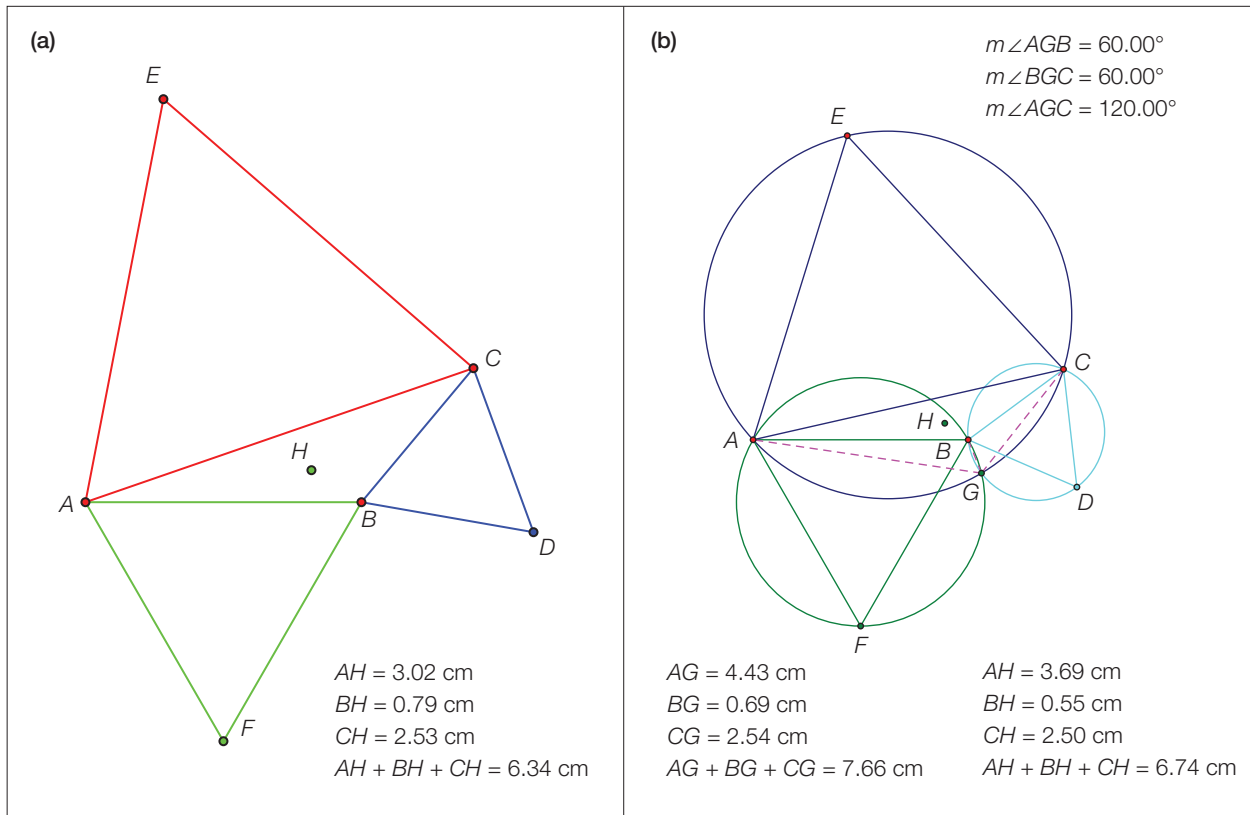$AH + BH + CH = 6.74$ cm

Figure 7. G is not the minimum distance point for some triangles.

At this stage of the investigation I ask students to characterize the triangles for which the equiangular point does not exist. A few claim that said point exists only for acute triangles while others realize that the equiangular point exists also for certain obtuse triangles (Figure 8a). To describe the distinctive nature of triangles for which the equiangular point does not exist, some students drag an appropriate vertex of the triangle (C in this case) to the extreme case where G still exists (i.e., "before" it coincides with vertex B) (Figure 8b). At this point most students realize that the equiangular point exists for triangles with no angle measuring greater than 120°. The class then formulates a refinement of conjecture 1:

Conjecture 2: The minimum distance point of a triangle with no angle measuring greater than 120° is the equiangular point.

The Final Conjecture

Our next task is to characterize the minimum distance point for triangles having an angle of measure more than 120°. To this end, students drag point H until the sum of its distances to the three vertices of the triangle seems to be minimized (Figure 9a, 9b, and 9c). At this point students hypothesize that the optimal point is the vertex of the obtuse angle. Further dragging for other triangles confirms our conjecture. Thus, our third, and final, conjecture is stated as follows,

Conjecture 3: The minimum distance point of a triangle a) is the equiangular point for triangles with no angle of measure greater than 120°, b) is the vertex of the obtuse angle for triangles having an angle measuring 120° or more.

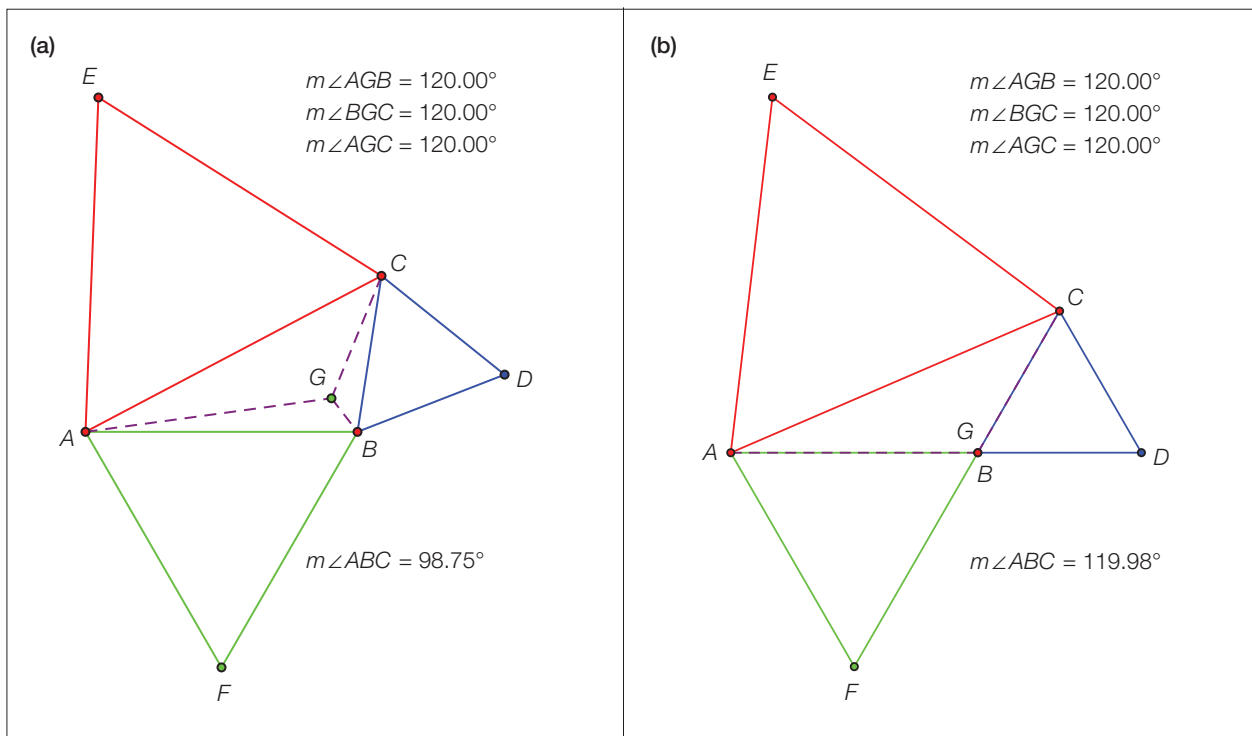*Figure 8.* Characterizing the triangles for which the equiangular point exists.

(a)

AH = 2.35 cm
BH = 0.38 cm
CH = 2.54 cm
AH + BH + CH = 5.27 cm

$m\angle ABC = 125.94°$

(b)

AH = 2.57 cm
BH = 0.00 cm
CH = 2.61 cm
AH + BH + CH = 5.18 cm

$m\angle ABC = 125.94°$

(c)

AH = 2.69 cm
BH = 0.28 cm
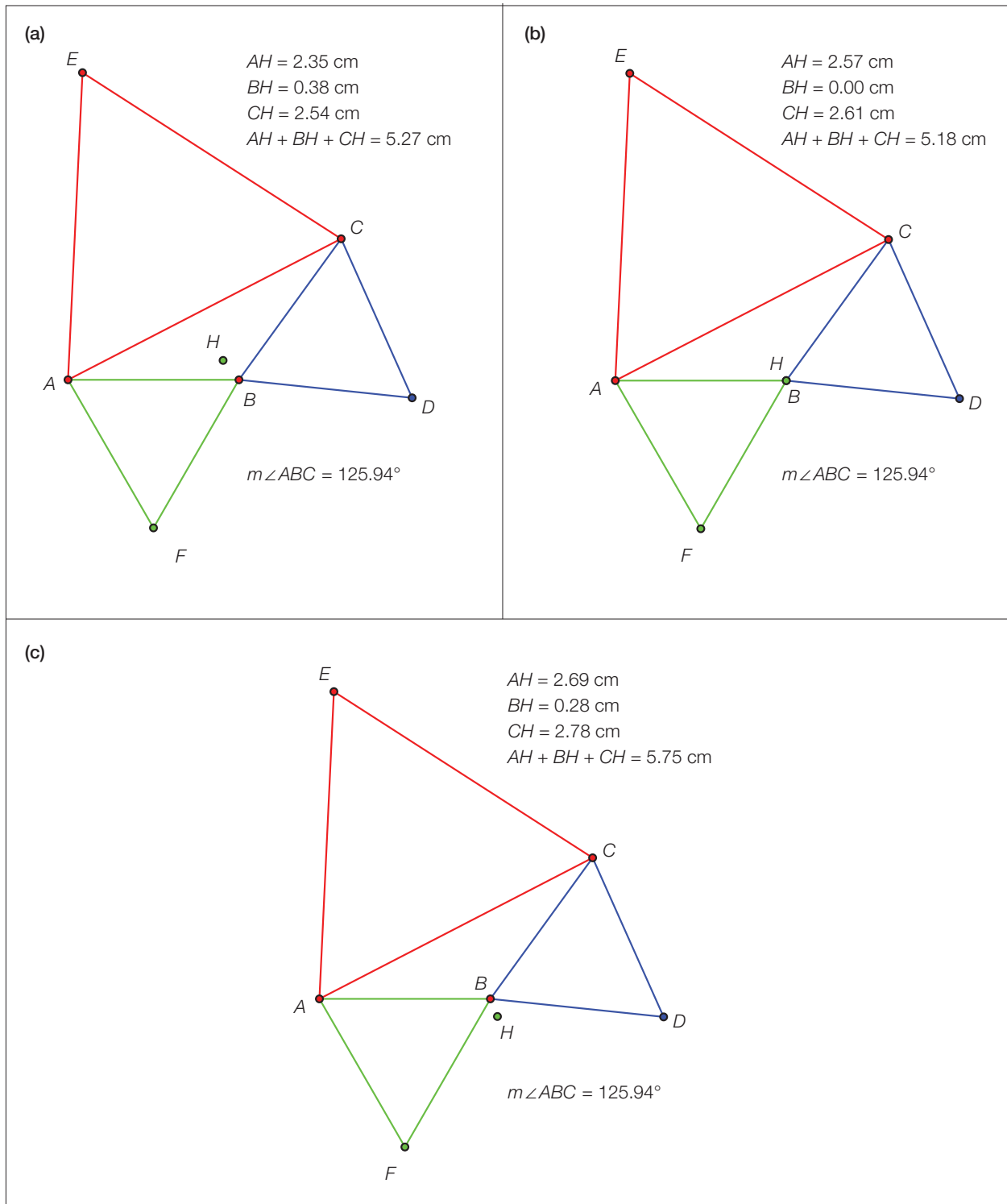CH = 2.78 cm
AH + BH + CH = 5.75 cm

$m\angle ABC = 125.94°$

Figure 9. Searching for the optimal point for triangles with an angle measuring more than 120.°

Reformulating the Conjecture as a Theorem

After the class formulates the final conjecture, I lead the class to develop an argument to prove that our conjecture is indeed a theorem. To this end, I ask students to construct the configuration displayed in Figure 10 with the following directions:

> Consider a triangle ΔABC with no angle measuring more than 120°. Let G be its equiangular point. Through A, B, and C, construct lines that are perpendicular to segments $\overline{AG}$, $\overline{BG}$, and $\overline{CG}$, respectively. Those lines determine ΔIJK.

To prove that G is the minimum distance point, a few students propose to consider another point, say L, and show that AL + BL + CL > AG + BG + CG. Notice that point L is an arbitrary point and that we are comparing L to G to show that no matter where L is, the sum of the distances to the vertices of the triangles is less than going from G than it is from L. At this stage students are "stuck" and I suggest that they examine ΔIJK. At this stage students determine that ΔIJK is an equilateral triangle because each interior angle measures 60° [for example, $m\angle BIC = 60°$ because $m\angle IBG = m\angle GCI = 90°$ and $m\angle BGC = 120°$]. I then ask students what property every interior point of an equilateral triangle has. After some moment of reflection, students realize that there is a connection between the fact that point L is an interior point of an equilateral triangle and Viviani's theorem: The sum of the distance from any interior point of an equilateral triangle to its sides is a constant. Using this theorem students then prove that G is the minimum distance point with the following reasoning:

> AL + BL + CL > NL + OL + ML (the length of the hypotenuse of a right triangle is greater than the length of any of the legs). Applying Viviani's theorem we obtain NL + OL + ML = AG + BG + CG. Thus, AL + BL + CL > AG + BG + CG.

To understand the restriction that ΔABC cannot have an angle measuring 120° or more, students drag point C

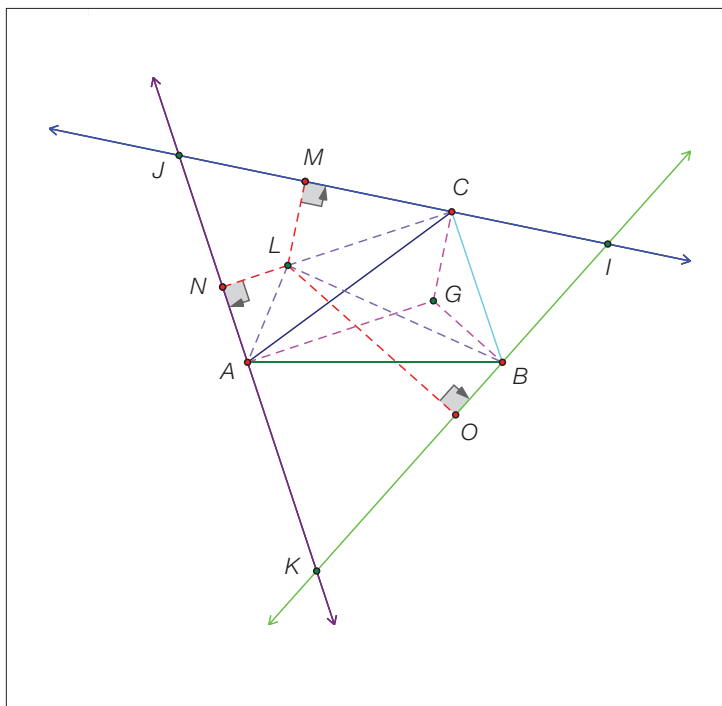*Figure 10.* Diagram to prove that G is the minimum distance point for triangles with all interior angles measuring less than 120.°

until $m\angle ABC$ is practically 120° (Figure 11). As they drag point C, they notice that the proof is valid for all intermediate triangles. The construction fails when we have $m\angle ABC = 120°$ and thus segment $\overline{GB}$ does not exist. As a consequence, line $\overleftrightarrow{KI}$ is undetermined. I then ask the class how we can reconstruct such a line so we can still have $m\angle BIC = 60°$ (that is, ΔIJK is still an equilateral triangle). After analyzing the situation, a few students propose to construct $\overleftrightarrow{KI}$ such that $m\angle CBI = 30°$. The proof that G (the vertex of obtuse $\angle ABC$) is the minimal distance point is still valid because Viviani's theorem is also valid for points on the triangle.

Students continue dragging point C to make $\angle ABC$ have a measure greater than 120° (Figure 12). They realize that the original process to construct ΔIJK is valid for triangles having an angle measuring more than 120°. The task is now to understand why point B, and not point G, is the minimum distance point. Students notice that G is an exterior point of ΔIJK and thus AG + BG + CG > ML + NL + OL = AB + BC. At this point, most students realize that the original proof can be adapted to show that B is the minimum distance point (AL + BL + CL > NL + OL + ML = AB + BC).
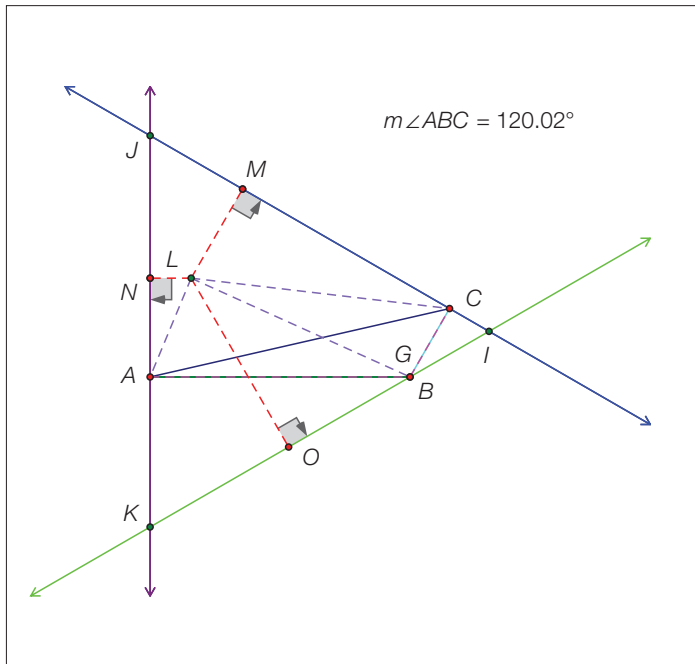
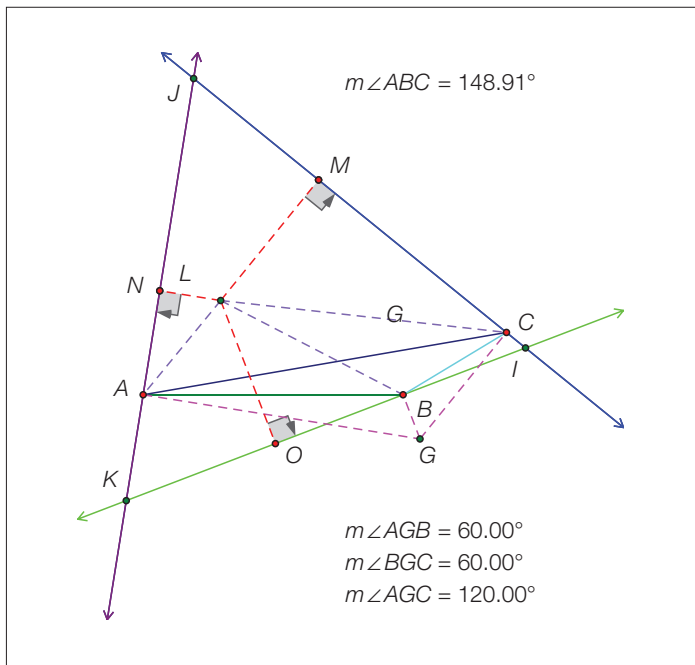*Figure 11.* The construction of ΔIJK fails for triangles having an angle measuring 120.°

Figure 12. A triangle with an angle measuring more than 120.°

Discussion and Concluding Remarks

The airport problem is an excellent example of how technology in general, and Dynamic Geometry Software in particular, can facilitate the process of solving complex mathematical problems. Based on visual and numerical feedback generated by GSP, students are able to generate, confirm, refute, and refine conjectures. By being asked strategic questions, students were able to use GSP to discover and characterize the two cases of the solution of the airport problem for triangles.

As is common when my students perform investigations within DGS environments, we justify our conjecture related to the solution of the airport problem with a formal proof. To date, there is still some debate about students' views about the need to prove a conjecture discovered and tested using the features of DGS (Furinghetti & Paola, 2003; Harada, Gallou-Dumiel, & Nohda, 2000; Heid & Blume, 2008). However, some pieces of research (de Villiers, 1998; de Villiers & Mudaly, n.d.; Hadas, Hershkowitz, & Schwarz, 2000; Jones, 2000; Mariotti, 2000, 2001; Sanchez & Sacristan, 2003) found that students are often motivated to understand why a conjecture discovered within a DGS environment is true. Once my students discover and formulate the final and complete conjecture for the solution to the airport problem, some express a need to further understand why the solution to the problems involves two cases: Empirical evidence is not only insufficient proof, but also does not provide enough insights to deeply understand the plausibility of a conjecture.

As students investigate the airport problem, they are engaged in a plethora of mathematical practices recommended by the Common Core's Standards for Mathematical Practice (CCSSI, 2010, pp. 6-8), including making sense of problems and persevering in solving them, constructing viable arguments and criticizing the reasoning of others, modeling with mathematics, using appropriate tools strategically, and looking for and making use of structure.

To start solving the airport problem, students need to understand its meaning, the given and unknown information as well as the constraints, which are components of the act of making sense of problems. Once they understand the problem they look for entry points to its solution. As the entry points are unproductive (using paper and pencil), they turn, sometimes not spontaneously, to the use of DGS to make an initial conjecture, which is refined as they continue looking for confirming and disconfirming evidence. Because the airport problem is not a straightforward problem, it provides students opportunities to further develop the habit of perseverance in solving problems. It is another example of the power of technology as a problem-solving resource.

Solving the airport problem involves gathering empirical evidence using Dynamic Geometry to formulate and refine a conjecture and then constructing a mathematical argument to justify it. In the process, students evaluate the arguments of their peers. As I described above, students initially proposed that the equiangular point of the triangle minimizes the sum of the distances to each of the vertices of any triangle. As students refine their conjecture, they continue to gather empirical evidence to support the new conjecture. Once students formulate the complete conjecture, I provide strategic hints to get them started in the construction of mathematical proofs to justify formally their final conjecture. They then modify and extend the original argument for the case when a triangle has an angle measuring 120° or more. During this process students evaluate the arguments developed by their classmates. They also visualize why the proof for the first case needs to be modified to justify the second case.

The airport problem affords students opportunities to model with mathematics. That is, they apply the mathematical principles they know (e.g., Viviani's theorem, the hypotenuse of a right triangle is longer than any of the other two sides, etc.,) to solve a problem that may arise in mathematics or in real life. Students are also able to analyze and identify the important mathematical features of the problem, how they are related, and how they lead to its solution.

The airport problem allows students to further develop their abilities to learn to use appropriate tools strategically, in this case Dynamic Geometry Software. Representing the problem with GSP allows us to gain insight about its solution which, in turn, leads to its discovery. The use of GSP facilitates the process of formulating, testing, and refining conjectures in ways that would be impractical, if not impossible, using only paper and pencil.

Lastly, but not least, the airport problem afforded students opportunities to look for and make use of structure. After students are provided with a strategic hint about how to construct ΔIJK and look for the relevant feature of point L in relationship to ΔIJK, they make use of this structural feature to use Viviani's theorem to further develop and complete the proof. As they are faced with the case of ∠ABC measuring 120°, they recognize the significance of reconstructing line $\overline{KI}$ such that ΔIJK is still an equilateral triangle so that Viviani's theorem can still be applied to this situation. As they consider the case when ∠ABC measures more than 120°, students again make use of structure to construct ΔIJK using the same initial procedure.

To conclude, using Dynamic Geometry Software facilitates the process of finding the unexpected solutions to optimization problems by ourselves. It allows us to experience the thrill of discovery, as no other teaching or learning tool does. Paraphrasing Movshovits-Hadar (1988), solutions to mathematical problems are an endless source of surprise. Certainly, the airport problem is not an exception!

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