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A View of Mathematical Modeling in Mathematics Education

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Mathematical Modeling is being introduced as a new product into the complex system of Mathematics Education. It has to fit with the existing parts and interfaces in this system. As one example, we will consider the effect of mathematical modeling on the transition from secondary to tertiary education. If mathematical modeling is of greater importance to the planners at one of these two levels of education than at the other, then stresses may result. As a second example, we propose to examine changes in teacher education necessitated by the introduction of mathematical modeling at the secondary level. Ideally, one might wish to prepare teachers to teach mathematical modeling by concentrating purely on modeling without the distraction of new mathematical ideas, but it is not clear that this can always be done. Finally, we plan to make some comments on the effect of mathematical modeling has on the relationship between mathematics education and mathematics itself. Each of these should gain from cooperation with the other.

Keywords: mathematical modeling, secondary education, tertiary education, COMAP

Introduction

When I was at Bell Laboratories, I learned the importance of thinking about the telephone system as a *total system*. Part of the job of Bell Laboratories was to make sure that any new product would work in the presence of everything that was already out there. Just one simple example: The first lightweight phones were developed in about 1959. From the point of view of the customer, it was *too* light-weight, and would not stand still when you used the rotary dial with one hand. Up until then, the phones were quite heavy. You might never have seen one except in an old movie. But you may remember the old Bell motto: You could have any color phone—as long as it was black.

So push button phones were developed in the early 1960s. They were great for the customers, but it became clear that a user would produce digits of a telephone number too rapidly for some of the existing telephone company equipment. We needed temporary storage of the incoming digits in the central office. All this gave a boost to the development of electronic switching.

In my second career, as mathematics educator, I see the same problem again, but with one enormous difference: There is no single organization whose responsibility it is to ensure that the whole system works together, and works well together as a system. The total system for mathematics education has many different components that have to work with each other, and many interfaces across which everything has to be compatible. We are now introducing a new product to many teachers in many countries: Some of us want to—though some of us have been told to—teach mathematical modeling. Now think of the parts of the system. Here are a few: There are curriculum planners; textbook publishers; textbook adoption procedures; teachers; the people who run the educational system, like principals, superintendents and commissioners of education. There are the customers, namely students; test makers; other disciplines that use mathematics; higher education; employers and parents, who are also customers; and politicians. Of course, we must not forget teacher educators—and who would dare to overlook mathematicians!

The effect of mathematical modeling on each of these system components is well worth studying, as is the effect of modeling on the interfaces between pairs of components. So the number of situations to be explored is somewhere between linear for the number of components, and perhaps as bad as quadratic in the number of interfaces between

pairs of components. This talk will take a look at just three of these potential sources of difficulty. They will be the interface between secondary and tertiary education, teacher education, and the interface between mathematics and mathematics education.

—Henry Pollak

First, I am not a fan of the Common Core State Standards of Mathematics (CCSSM), but they are here and their implementation, for better or for worse, will occupy a central place in US mathematics education for at least a decade. There are many reasons why I am unhappy with the standards. The first is the fact that the Practices take up 3 pages and the laundry list of topics 90. I also dislike the concentration on arithmetic fluency in the early grades and the movement from descriptive geometry to transformations with insufficient time spent on the usefulness of these ideas.

Statistical and probabilistic ideas are way too far delayed and from my point of view they demonstrate a poor understanding of the modeling process with a fixation on fitting curves and functions to data. There is, as well, a paucity of discrete mathematics both early and late, specifically as it applies to any number of application areas. So mea culpa, I am working to make the CCSSM implementation go as smooth as possible. For instance, I serve on the Partnership for Assessment of Readiness for College and Careers (PARCC) Content Working Group.

I do want to commend the writers of the new National Council of Teachers of Mathematics (NCTM) publication . . . I believe that unlike the Common Core this document really does represent an evolution of the 1989 NCTM Standards and moreover it puts leadership in mathematics education back where it belongs—in the hands of the profession!

Well, where is modeling in all this? As we've said, modeling is one of the eight Practices listed in CCSSM. An (I believe) unintended consequence of this fact is that COMAP has received a large number of requests for modeling materials. In particular, from teachers interested in a better understanding of what modeling looks like in practice. This has led to a good deal of interest in the Modeling Handbooks. It also represents new life for COMAP materials, both for curriculum and for staff development. Interest has been expressed by some publishers in adapting some of our materials for new delivery modes, e.g., tablets.

In part this is a matter of timing. The standards say modeling, but the books aren't there and the tests haven't been written. So teachers are excited and somewhat scared and want to be prepared. Hence they want us to show them examples of modeling and what and how they should teach it.

It is an educated guess that when the Programme for International Student Assessment (PISA) 2012 results are announced and analyzed, US deficiencies in the higher order thinking necessary for mathematical modeling processes may very well be highlighted, especially as compared with typically high performing countries such as China (Shanghai). This will likely bring renewed interest and activity. Also, it is a fair guess that when we see the results of the performance-based modeling tasks on Smarter Balance and PARCC items, they will be dismal, which may result in pressure for additional class time emphasis.

—Sol Garfunkel

The Effect of Modeling on the Relationship Between Secondary and Tertiary Education

Modeling in High Schools—What Have Chinese Universities Done?

In the United States, there is a major effort to introduce mathematical modeling into the schools. Will the universities change their placement examinations to take this new ability into account? College placement examinations have the reputation of being among the most immovable features of our mathematical scene. And how will students' ability to model mathematically be used at the undergraduate level? Will calculus courses, will elementary science and economics courses, keep this new ability alive, never mind actually use it to good effect?

Will high schools be expected to put modeling into AP courses? A first primitive reaction: If you don't, then they are not high school courses; if you do, then they are not college courses.

Modeling in Universities—What Will High Schools Do?

The opposite situation could become a problem in China. On the one hand, in recent years, there has been a rapidly increasing interest in mathematical modeling at the tertiary level—see, e.g., Tian and Xie (2013). On the other hand, the history of the National University Entrance Examination in mathematics gives the impression that it, too, is a rather immovable feature on China’s mathematical scene. Will the national examination change in the direction of inclusion of mathematical modeling, and then will the secondary curriculum reflect such a change? If modeling has become so important at the tertiary level, can the secondary preparation ignore mathematical modeling?

Looking at the developments in China we see incredible growth. As many of you know, COMAP administers the Mathematical Contest in Modeling. When the contest began 30 years ago it had 90 teams from 70 US colleges participating. Last year there were 6,700 teams from 20 countries, 6,200 of those teams from China alone. The Chinese have made a major commitment to modeling at the undergraduate level. Every math major and every future teacher is required to take a modeling course! The growth has been nothing short of phenomenal.

However, this is not yet true at the high school level and the following is a cautionary note about high stakes testing. The Chinese have an end of high school test, which is high stakes indeed. Results on this test determine whether students go to university and if so what level of university they are permitted to attend. It literally determines their life course. As you can imagine, students are completely focused on doing well on this test. At this point, the test is quite conservative by our standards and in particular contains no modeling. Hence, it is almost impossible to introduce modeling into the Chinese high school curriculum.

Our university colleagues are working hard to change that and are working on helping establish an international modeling competition at the secondary level, which can elevate the importance of modeling at this level. COMAP hopes to work with them in this effort. We also firmly believe that the time is right for a real sea change in the teaching and learning of mathematical modeling in the US, and from the standpoint of international competitiveness, it doesn’t come a moment too soon.

One final comment about the United States: If we are successful in teaching modeling in secondary school, then the problem will be to keep it alive and to use it at the tertiary level. If we fail in teaching modeling at the secondary level, then others will *have* to teach it at the tertiary level.

Teacher Education—Preparing Teachers to Teach Mathematical Modeling

Teaching About the Modeling Process: Can it Involve New Mathematics?

We in the United States have an enormous job preparing in-service teachers to teach mathematical modeling. What are some of the issues? First of all, teachers must see, or better still, they must work through, sample materials that teach mathematical modeling. They won’t know what you are talking about without that. What is involved in taking a group of teachers through a modeling experience? It means that they must participate in formulating the problem situation, deciding what to keep and what to ignore in creating an idealized model, doing the mathematics in the idealized situation, and then examining whether or not the results make sense in the original situation. A single example of this might take longer than a typical classroom period, which may cause a scheduling problem. Textbooks may not facilitate the need to spend several class periods on one modeling problem. In fact, what the textbooks do with the Common Core State Standards for Mathematics modeling mandate is very interesting, and we have a Teachers College graduate student working on related issues right now. A different kind of difficulty, teacher educators might tell you, is that the mathematics involved in the idealized situation should be completely familiar to the teachers. The argument is that you do not want to distract from teaching the modeling process by teaching new mathematics at the same time! You do not want to distract the teachers from the new thing, namely mathematical modeling. And yet the very fact that this mathematics came from a modeling situation may naturally lead to mathematical questions that you *would not have asked* otherwise. This is a serious difficulty, and we will return to this problem when we will look at an example of a modeling situation.

Assessment. How to Judge the Success of a Model?

It is natural that teachers' comfort with mathematical modeling will be facilitated by seeing what assessment questions for modeling will look like. Teachers want to know how the material they are teaching is going to be tested. But beyond that, assessment has been very much in the news, and has become of significant and even international political interest. But, as we see in every context we examine, mathematical modeling brings a whole new dimension into assessment.

Think about what it means to assess success in modeling. The idealized model must give results that are mathematically correct. Fine. But for the first time in the experience of many teachers, *mathematical* correctness is not enough. The results must also be sensible within the situation that is being modeled: The results must be sensible from *both* points of view. This is a new experience for many teachers. If you want to be pompous about it, it is a loss of sovereignty for mathematics as it has been taught: that is, for unapplied, pure, mathematics. How will teachers, and other levels of the mathematics education system, react? By the way, how will mathematicians react?

About 40 years ago, I was an invited guest at a national summer conference whose purpose was to grade the AP Examinations in Calculus. When I arrived, I found myself in the middle of a debate occasioned by the need to evaluate a particular student's solution of a problem. I wish I could remember all the details, but I do not. The problem, as best as I can remember, was to find the volume of a particular solid, which was inside of a 3-dimensional unit cube. The student had set up the relevant integrals correctly, but had made a computational error at the end and come up with an answer in the millions. (I think he multiplied instead of dividing by some power of 10.) The two sides of the debate were as follows: (1) He set everything up correctly, he knew what he was doing, he made a silly numerical error. Let's *take off* a point. (2) My God, he must have been sound asleep! How can a solid inside a unit cube have a volume in the millions? It shows no judgment at all. Let's *give* him a point.

My recollection is that side (1) won the argument by a large margin. But suppose that you have allowed modeling to invade AP Calculus. Then it would no longer be an argument just from the traditional mathematical point of view. As I said, in a mathematical modeling situation, pure mathematics loses some of its sovereignty. The quality of a result is judged not only by the correctness of the mathematics done within the idealized mathematical situation, but also by the success of the confrontation with reality at the end. If the result does not make sense in terms of the original situation in the real world, then the student has not confronted reality at all. Now how would you vote?

I need not remind you of one of the topics that has been much debated in recent years: Is the main purpose of teaching modeling to help motivate and to help teach the pure mathematics in the idealized model, or is our main purpose to teach mathematical modeling for its own sake? Peter Galbraith has used the wonderfully compact phrases "modeling as vehicle" versus "modeling as content." Is our purpose really one of these two, or is it a mixture of both? Will the various players in the system of mathematics education agree on the relative importance of modeling as vehicle and modeling as content? One of the pieces of the mathematics education system which I said I was *not* going to talk about today is the planners of the ideal curriculum, but surely this affects them.

What Do Teachers Believe Modeling Is?

If you are going to work with the mass of teachers who are going to be teaching mathematical modeling, then there is an interesting set of questions that you really should consider before you start: What do teachers believe is the meaning of the phrase "mathematical model?" Is a map a mathematical model? Is an architectural blueprint a mathematical model? Is that dodecahedron in the display case in the hall a mathematical model? And what do teachers think mathematical modeling means, and what do they think the purpose of teaching mathematical modeling is? Answers may be found in "Teachers' Conceptions of Mathematical Modeling" by Heather Gould (2013). Here are brief statements of some of Gould's results:

Mathematical Models

- Almost all teachers believe that mathematical models can be physical manipulatives such as fraction tiles, pattern blocks, and 3D solids.
- Teachers agree that mathematical models can be objects such as maps and blueprints.

Mathematical Modeling

- Teachers believe that mathematical modeling situations can come from whimsical, unrealistic scenarios. Arthur Engel's kingdom of Sikinia has some wonderful stuff, but is it modeling?
- The majority of teachers do not believe that making choices and assumptions is always part of the modeling process.
- About a third do not believe that you always have to check the mathematical solution in the context of the modeling situation with which you began.

Mathematical Modeling in Education

- Teachers are much surer that the curriculum *intends* modeling as vehicle than they are that the curriculum intends modeling as content. Only about a third are actually confident of the latter. As an example of their own beliefs, they strongly believe that modeling helps in understanding scientific phenomena, but only a third see modeling as helping in the social sciences and the humanities. Apparently, topics such as fair division, apportionment, and elections have not reached many teachers.

The Relationships Between Mathematics and Mathematics Education

A Modeling Problem

The following modeling problem is readily accessible to students: You are a passenger in a car, and you wish to estimate the speed of a vehicle that is going in the same direction as you are, and is passing your car.

The scene is a divided, multiple lane highway. First of all, make it very clear that the situation is that you are a passenger in the car, not the driver. We don't want drivers to think about mathematics rather than about their driving. The first part of the modeling situation is the following: "As a second car starts to pass the car in which you are a passenger, suppose you decide you would like to estimate how fast the other car is going." What kinds of approaches might students take? You are in car A , the passing car is car B . You know the speed, $v(A)$, of the car you are in. You see a landmark ahead, a bridge or a road sign for example, and you count the time, $t(B)$, it takes for car B to get there, then compare it to the (presumably longer) time $t(A)$ for your car to get there. You know that distance equals velocity multiplied by time for both cars, so you know that $v(A)t(A) = v(B)t(B)$. Now your estimate for $v(B)$ is $v(A)t(A) / t(B)$. Your first act as a modeler is to check that this formula makes basic sense: $t(A) > t(B)$, so $v(B)$ will be greater than $v(A)$; so far, so good.

Now you have a formula. What assumptions did you make to obtain this formula? That both cars keep going, each at a constant speed. How accurate can you expect this computation to be? Here comes the first surprise: If your counting of seconds is not correct, but either too fast or too slow, what will this do to the answer? This will take some time and discussion, because this type of question is probably new to the students. The answer is, it does not matter! Your velocity estimate is not affected. This is a property of homogeneous linear functions, or, if you prefer, of a proportion. Of course if your counting is uneven, that *will* affect your answer. Another aspect of accuracy: Do the syllables in the language in which you count time allow you to estimate half seconds? This will help your accuracy. Is there another method to estimate time? Most people do not carry a stopwatch. Perhaps a student will make the following interesting suggestion: It is raining. Use the windshield wipers to measure time.

Think back to our earlier discussion: Yes, this problem requires only the most familiar of mathematics, but the discussion of accuracy will probably be a new aspect of the pure mathematics in your model. The modeling situation demands that you examine an aspect of homogeneous linear functions, or of proportionality, that perhaps does not arise in the teaching of the pure mathematics! And to return to another previous question: Is this modeling as vehicle (no pun intended) or modeling as content?

Another part of the modeling process: What do you mean by the time that either car reaches a landmark? Does it matter whether you try to time the front of the car or the rear of the car when it gets to the bridge? Will that affect your answer more or less than the accuracy of time to at best a half second? Does curvature in the road matter? How about the extra time it takes the passing car to change lanes? Does it matter? Those kinds of questions are part of modeling. What do you have to keep, and what can you afford to ignore?

Now comes the second part of the model: What if you do not decide right away that you want to estimate $v(B)$? Can you still make an estimate *after* car B has passed car A ? Perhaps the students will think about using two landmarks instead of one. What happens to the sensitivity of the model? What, if anything, changes in our previous modeling decisions?

Next, a third part of the model: In the United States, the longest truck permitted on the highways as of this writing is 53 feet. Suppose your car is being passed, but passed slowly, by such a long truck. You can count how many seconds it takes the whole truck to pass car A . That's another way to make an estimate of speed. Add the rate at which the truck is passing car A to the speed of car A , and you have the speed of the truck. How does this differ from the previous method? Is the method still insensitive to the accuracy of your counting? (Answer: No.) And you will have to know how to convert between the feet and seconds in your measurement of the truck and the miles per hour of $v(A)$. And, is the 53 feet the length of the trailer, or is it the length of the trailer plus the cab? What does the 53 painted on the side of the van actually *mean*?

If both methods are available to you, how would you choose between them? Here is one possibility: If the truck is only a little faster than car A , then the new method will do better; if the truck is much faster, then you are better off using a distant landmark. Not many students may get that far in the discussion. Is there a possible optimal combination of the two methods of estimating? That sounds like a more difficult problem.

A potential assessment problem for the preceding modeling experience: What if you want to estimate the speed of an oncoming car, a car that is going in the *opposite* direction? If you can find a nice model for this, it might be simpler or more general than the one we considered. You don't need a divided super-highway for that; you can do it on an ordinary two-lane road. But perhaps this will introduce other complications.

Modeling as Vehicle and Modeling as Content

The mathematics used in the preceding idealized model is very simple and very familiar, but some of the questions about sensitivity of the mathematical results may *not* be very familiar. If you are giving a course on the *teaching* of mathematical modeling, perhaps you cannot always avoid doing some new mathematics.

During the 2011–2012 academic year, a team of students at Teachers College, in cooperation with COMAP, the Consortium for Mathematics and its Applications, prepared a handbook consisting of 26 modules, each of which consisted of a modeling problem ready for classroom use at the high school level. The handbook became available in the summer of 2012. Last fall, Heather Gould and I ran a Saturday workshop on the teaching of mathematical modeling: six half days during which we planned to teach 12 of the modules. We had a group of 31 people, partly current secondary or college teachers of mathematics, and partly mathematics education graduate students at Teachers College. We ended up teaching 7 of the 12 modules, but only by combining two of them into one session: It took twice as long for the group to complete a module, roughly four hours rather than the two, then we had expected. Professor Margaret Kidd, who had tried some of the modules working with in-service teachers in California, had warned us that this is exactly what would happen! Even though most members of the group were participating in the workshop voluntarily and were not taking it for university credit, almost all stayed with us for all six sessions. (And one of the sessions was the Saturday after Hurricane Sandy!)

A second TC-COMAP handbook was prepared in the 2012–2013 academic year; namely, a handbook of assessments for each of the 26 modules in the first handbook.

Primary and Middle School

When it comes to elementary arithmetic, there are some aspects of the elementary operations that students may not learn in elementary school. If you add two numbers, A and b , where A is large and b is small, the precision of b , and probably b itself, do not matter at all. On the other hand, if you subtract B from A , where A and B are almost equal, the answer may be meaningless. Dividing by 0 is forbidden, but dividing by almost 0, while not forbidden, is probably stupid. These instincts should be acquired in elementary school. A very good way to acquire them is from modeling experiences.

Twenty-five years ago there emerged a path-breaking experimental project called “The Regional Math Network” lead by Kay Merseth from Harvard, with support from the National Science Foundation, which produced four volumes centered on—sports, ice cream, the Quincy Market in Boston, and outer space—and which used a huge variety of applied and modeling situations to teach seventh and eighth grade mathematics. But what was so

exceptional is that it took full responsibility for the mathematics traditionally taught at that level. It didn't just motivate, or just apply, it guaranteed to do it all. It was way ahead of its time.

Secondary School

COMAP's "Mathematics, Modeling Our World" series was created in a similar spirit. As suggested above, mathematical modeling should sometimes endeavor to take responsibility for a particular mathematical topic: Create the necessity for it, develop it, and fit it into the system. You don't want the *body* of the mathematics education system to reject something that feels like a foreign invader; you want it to absorb a new part of itself.

The Secondary to Tertiary Interface

One more example that I can't resist mentioning: In the traditional curriculum, we teach complex numbers in grade eleven as a part of the work with quadratic equations, but then complex numbers are not mentioned again until second year calculus, when students are ready to learn about e and then e^i . But you could do a discrete version of an oscillating spring using just DeMoivre's Theorem in high school rather than complex exponentials, and get a nice "modeling as vehicle" example.

Coda

Where do we go from here? I believe that modeling problems should be problems from the real world, and I believe we can teach such problems. There are good problems all around us: many people, including myself, put a lot of effort into finding such problems. But I believe that teaching a true modeling problem takes time. I have seen examples of mathematics education systems where you cannot find a whole period, never mind a week, for students to discuss a modeling situation, formulate an idealized model, do the relevant mathematics, and then examine the success of what we have accomplished. We must avoid mathematical modeling getting a reputation as just a fancier terminology for the same old word problems. We must somehow find the time it takes to go through the complete modeling cycle. Probably not every time, but can we have three or four hours every few months during which to do full-scale modeling? This may be a battle within the mathematics education system, but in my opinion it must be undertaken.

A number of people have written books entitled something like "The Joy of Mathematics." I should like to see a book entitled "The Joy of Mathematical Modeling," consisting of fifty to a hundred examples, taken mostly from everyday human experience. The joy I have had in my life of doing and teaching mathematical modeling should be transmitted: Will you join me?

References

- Fletcher, N., Velamuri, A., Waid, B., & Dimacali, A. (Eds.). (2013). *Teachers college mathematical modeling handbook ii: Assessment of mathematical modeling*. Bedford, MA: COMAP.
- Gould, H. (2013), "Teachers' Conceptions of Mathematical Modeling," Doctoral Dissertation, Columbia University, 2013.
- Gould, H., Murray, D. R., & Sanfratello, A. (Eds.). (2012). *Teachers college mathematical modeling handbook*. Bedford, MA: COMAP.
- The Regional Math Network, Kay Merseth, Director, Harvard Graduate School of Education, 1987.
- Tian, X. & Xie, J. (2013), Summer camp of mathematical modeling in China. *Journal of Mathematics Education at Teachers College*, 4(2), 39–43.