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Mathematical Modeling: How Can Students Learn to Model?

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This paper deals with the learning and teaching of mathematical modeling. After a definition of modeling competency, some empirical results about students' learning of modeling will be reported. After that, five criteria for quality teaching of modeling will be discussed. Finally, a modeling unit for the ninth grade will be presented, as well as some results of the accompanying research.

Keywords: mathematical modeling, modeling competency, independent learning, quality teaching, cognitive activation, teacher feedback

Introduction

The title of this paper—as suggested by the symposium organizers—addresses learning, but it will turn out that its content will be more about teaching, because the short answer to “How can students learn to model” is not simply “by doing it independently,” but “by quality teaching!” In this context, “learning to model” means acquiring modeling competency.

The paper will comprise four parts. First I will address the notion of modeling competency. Second, the focus is on the learning of modeling: What do we know empirically about students' independent modeling activities? In the third part, the question arises as to how these activities can be supported by quality teaching. The fourth and final part presents a modeling unit for the ninth grade that we have developed, as well as some results of the research that accompanied its creation.

Mathematical Modeling Competency

As a concrete introductory example, I will use the “Filling up” task (Blum & Leiß, 2006) to which I will refer several times (Figure 1). It is a modeling task designed for ninth grade students.

Filling up

Mister Stein lives in Trier, 20 km away from the border of Luxemburg. To fill up his VW Golf he drives to Luxemburg where immediately behind the border there is a petrol station. There you have to pay 0.85 Euro for one litre of petrol whereas in Trier you have to pay 1.1 Euro.

Is it worthwhile for Mister Stein to drive to Luxemburg? Give reasons for your answer.



Figure 1.

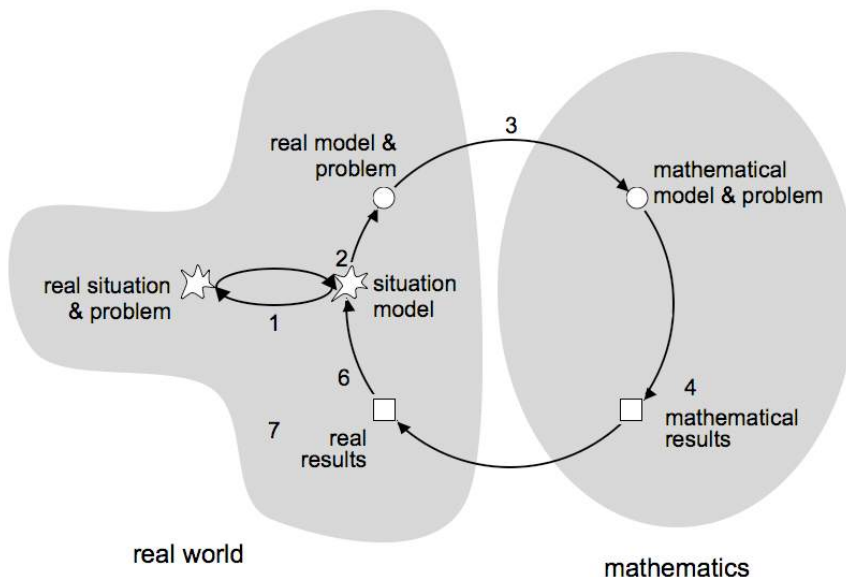
The task is about a certain Mr. Stein—a real person—who lives in Trier, which is close to the border with Luxembourg, and who drives to Luxembourg in order to fill up his Volkswagen. You can see from these numbers that we designed this task about ten years ago.

This is a real world situation and, according to Henry Pollak’s famous maxim, “Here is a situation, think about it” (Pollak, 1969), we will analyze this situation, and as a tool we will use the version of the modeling cycle proposed by Blum & Leiß (2007) and seen in Figure 2.

The numbered steps can be summarized in the list below.

1. Constructing
2. Simplifying/ Structuring
3. Mathematising
4. Working mathematically
5. Interpreting
6. Validating
7. Exposing

Figure 2.



This seven-step version shows an idealized typical way in which the relationship between mathematics and the real world, or according to Pollak (1979) the “rest of the world,” is. In our example, the cycle starts with the task, which is now a section of the real world. The first step is to build a mental model of the real situation: There is Trier, and there is some road to Luxembourg; there is a gas station behind the border, and we have to structure, to simplify, and to idealize the situation in order to make it mathematically accessible. Thus, we build a real model of the situation, and in this first approach we are only interested in the price and nothing else. As a result, we have two prices and can make some assumptions: Fifty liters for the volume of the tank, and eight liters per hundred kilometers for the consumption. We are now prepared to build a mathematical model (Figure 3).

We carry out some calculations and find a difference of 9.78 Euros. We interpret this result to mean that Mr. Stein saves approximately 10 Euros per tank of gas if he drives to Luxembourg in order to fill-up his car. We round it off since, as John Maynard Keynes famously quipped, it is better to be roughly right than precisely wrong. The next important step is the validation of the result: What have we ignored? The air pollution, the time, the risk of an accident, etc. Who is Mr. Stein? Does he have the time to drive to Luxembourg? A lot more variables come into

$$C_{\text{Trier}} = 1.1 \text{ €} \cdot 50 \text{ l}$$

$$C_{\text{Lux.}} = \left(0.85 \text{ €} \cdot 50 \text{ l} \right) + \left(0.85 \text{ €} \cdot 40 \text{ km} \cdot \frac{8 \text{ l}}{100 \text{ km}} \right)$$

$$S = C_{\text{Trier}} - C_{\text{Lux.}}$$

Figure 3.

play, thus, we may choose to refine the model and go around the loop illustrated earlier several times until we are satisfied. Ultimately, we write down our solution.

This seven-step cycle of the modeling process is a blend of models that come from cognitive psychology, especially the mental model, from linguistics, and from applied mathematics, where usually the step from the situation to the mathematics is much shorter.

Provided the background above, we define *modeling competency* (see Blum, Galbraith, Henn & Niss, 2007) to mean an individual's ability to construct and to use mathematical models by carrying out appropriate steps according to the problem, and/or analyzing or comparing given models. We can regard the seven steps in the cycle as corresponding to modeling *sub-competencies* (see Maaß; 2006; Kaiser, 2007).

What Do We Know About Students' Independent Modeling Activities?

Based off of this question, we can pose another: What are students able to do if they are confronted with such problems and can work on them independently? Of course, if we see the number of steps that are necessary, it is clear that modeling is a cognitively demanding activity. It involves several competencies (Niss & Højgaard, 2011) including reading a text and all the other steps in the modeling process. In addition, real world knowledge is necessary, along with a lot of procedural and conceptual mathematical knowledge. As a result, it is no wonder that modeling is empirically difficult as evidenced every three years by the results of the PISA tests. Furthermore, we know empirically that each step of the modeling process is a potential *cognitive barrier* and can be a source for getting stuck or for making mistakes; it is, in the words of Peter Galbraith and his group, a potential blockage (Galbraith & Stillman, 2006). Or as Hugh Burkhardt and his group in the Shell Center expressed it thirty years ago, "The weakest link in the modeling chain will set the limits on what they can do" (Treilibs, Burkhardt & Low, 1980).

Let us take a closer look at the first step: "Understanding the situation and constructing a situation model." Figure 4 shows a student's solution to the "Filling up" problem; one does not need to understand the German in order to see what was carried out.

This is a typical solution with regard to the maxim: "Ignore the context, just extract all data from the text, and calculate something according to a familiar schema." This is a very well-known substitute strategy all around the world, as well as in Germany, and I know it is also very popular in the United States. This is often called the "suspension of sense making in the word problem game" (Verschaffel, Greer & DeCorte, 2000). Here are three more examples where the answers are often given by "proportional reasoning":

- The 100 m world record for men is 9.58 sec. What is the 1000 m world record?
- 2 eggs take 5 minutes to get boiled. How long will 9 eggs take?
- Henry VIII had six wives. How many did Henry IV have?

The second step of "simplifying, structuring, idealizing" can also provide a cognitive barrier. Here is a solution of a grade nine student to the "Filling up" problem (translated):

You cannot know if it is worthwhile since you don't know what the Golf consumes.

You also don't know how much he wants to fill up.

The student stopped here. He obviously understood the situation and identified the relevant variables, but then did not make any assumptions. This is a very clear outcome of a lot of studies: Students do not like to make assumptions by themselves. They are used to problems where all the necessary information is already given in the problem statement.

The sixth step of "Validating" is difficult to observe in a written solution, but often

20: 0,85 = 23,53
 20: 1,1 = 18,18
 Nein die Fahrt lohnt sich nicht, denn wenn Herr Stern nach Luxemburg fährt dann hätte er schon allein für die 20 km 23,53 pro Liter gezahlt. Dann noch zurück zwar mit vollem Tank! Wenn er nach Trier fährt hat er einen kürzeren Weg zurückzulegen

Figure 4.

there is no validation at all. It is part of the silent didactical relation between teacher and students: Only the teacher is responsible for the correctness of the solution, not the students themselves.

Another observation is that normally students do not seem to have *strategies* available, and they frequently do not reflect upon their solutions nor are they able to transfer them, even to a structurally similar situation. If, for instance, the situation is not about driving with a car to fill up, but the question is whether or not it is worthwhile to drive to a department store in order to buy something because it is cheaper there, students consider this as a new task. Hence this is a special instance of situated cognition, that is, everything you learn, you learn in specific contexts, and transfer does not take place unless it is organized. Transfer is especially difficult for modeling; so what can “modeling competency” signify? Perhaps there is no such thing as “general modeling competency.” Modeling competency always carries with it, so to speak, the indices of the context in which it was developed. For example, in our project CO²CA (“Conditions and consequences of classroom assessment”) it turned out that the correlation between modeling in an environment where the Pythagorean Theorem was the mathematical background and modeling where linear functions were the background is only 0.6.

Another observation in several studies is that students usually do not follow this ideal-typical seven-step cycle. This is a special instance of Rita Borrromeo Ferri’s *modeling routes*, also dependent on the thinking styles of students (Borrromeo Ferri, 2007). Normally there is some jumping forth and back, some mini-loops occur, and sometimes students get stuck along the way.

How Can We Support Students to Acquire Modeling Competency?

This is the core question, and the obvious answer is: by quality teaching. We know a lot about quality teaching from empirical research and also from theories on teaching and learning (for an overview see e.g., Blum, 2011). Here are five aspects with fuller explanations below. These aspects are very much related to what Alan Schoenfeld pointed out in his presentation (see page 13), though they are listed under different headings. In all cases, the criteria are necessary but not sufficient.

1. Effective and learner-oriented classroom management; this is more or less related to surface structures but also important.
2. Cognitive activation of the learners. To be more precise, of all learners.
3. Meta-cognitive activation of the learners.
4. Competency-oriented orchestration of topics; students must have the opportunity to actually do all these activities; so this refers to the mathematical substance of it, the cognitive demand of the mathematics involved.
5. Appropriate feedback.

A.E.1

The *classroom management* (e.g., using time effectively, separating learning and assessment recognizably, using students’ mistakes constructively, or varying methods and media flexibly) is mostly content independent or subject independent. Many studies that have tried to find out which factors contribute to effective learning have shown that appropriate classroom management is a necessary condition. Group work is especially suitable for modeling and, more generally, a mixture between individual work, partner work, group work, and whole class work will be appropriate.

A.E.2

It is necessary to *stimulate students’ own activities*. Modeling is not a spectator sport, as has been said several times before, but rather must be done by the students themselves. An important distinction is between students working *independently*, and students working *alone*. Independent work means that the teacher is available, if necessary, and tries to let students work as independently as possible, but not to leave them alone in the desert. A key aspect, which sounds trivial but is in some sense the key to effective teaching, is always to seek a balance between students’ independence, on the one hand, and teacher’s guidance, on the other hand. This is the so-called *principle of minimal support*, which was perhaps first formulated by Hans Aebli, a Swiss pedagogue and one of the students

of Jean Piaget (Aebli, 1985). A key concept here is *adaptive* teacher intervention, i.e., an intervention which allows the individual to continue his/her work independently, which helps him/her to overcome a cognitive barrier but not more than that, and, in particular, does not prevent mistakes before a cognitive hurdle is even presented. Of course, whether an intervention is adaptive or not can only be judged afterwards: Did the student overcome the hurdle? If so, it could have been adaptive or not. If not, it was certainly not adaptive. Let us turn next to another example in the context of the “Filling up” task. A lot of students make a mistake by assuming the distance traveled is twenty kilometers, forgetting that the drive back needs to be accounted for. A simple but often successful intervention is to say, “Imagine the situation concretely. Imagine you are Mr. Stein and you drive to Luxembourg.” This strategic intervention often seems to be adaptive as students can discover their mistake by themselves. Here is a list of possible strategic interventions that ought to belong to the repertoire of the teacher, and for which the teacher must decide which items are appropriate and when they should be used.

- Read the text carefully!
- Imagine the situation clearly!
- Make a sketch!
- What is your intention?
- What is missing?
- What data do you need?
- How far have you gotten?
- How far are you from the solution that you are aiming for?
- Does the result make sense?

If this not sufficient, then of course more content-related interventions can be appropriate. Empirical studies show that in everyday classrooms there are nearly no strategic interventions.

A.E.3

Cognitive activation is not enough. We know from a lot of studies that *meta-cognitive activation* is equally important (Schoenfeld, 1994; Burkhardt & Pollak, 2006; Stillman, 2011) especially if we hope for transfer. Transfer does not occur by itself, and in order to have some transfer, it is necessary to switch to the meta-level and to make students aware of what they are doing, with accompanying or retrospective reflections. A promising approach is to try to advance *learning strategies*. Strategies on an intermediate level are still not task-specific but specific for certain types of tasks. Here is the strategic instrument that we have used in the DISUM project for ninth-graders, a four-step modeling cycle (“Solution Plan”; see Blum, 2011). Seven steps are too complicated to use, but four steps are appropriate for students:

Understanding the task → Searching for mathematics (that means building a mathematical model, comprising steps two and three in the seven-step cycle) → Using the mathematics → Explaining the result (comprising steps five to seven in the seven-step cycle).

If this is not satisfactory, then the cycle will start again. This is something that we gave to teachers as a tool for their interventions and support, and to students for helping them in the solution process; it will be returned to later on.

A.E.4

Competency-oriented orchestration of the topics. This means students need to have the opportunity to practice their desired competencies. Modeling is only learned by modeling, arguing by the arguing, and so forth. Important, too, are links between topics, vertical and horizontal links, as well as intelligent practicing. As noted before, no transfer can be expected. Modeling can be learned best by modeling activities, and this is a long-term learning process beginning with early implicit models, and continuing indefinitely with repetition and practicing. Some teachers believe repetition and practicing are old-fashioned; however, it is clear that our brains need repetition and practice in order to effect noticeable change.

A.E.5

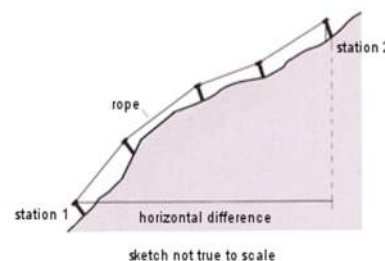
An important quality aspect is appropriate *feedback*. According to Hattie & Timperley (2007) we have to have a permanent connection between a sound diagnosis and stimulating feedback and support for students by appropriate interventions. I will refer here to some results of the CO²CA project (“Conditions and Consequences of Classroom Assessment”; see Besser, Blum & Klimczak, 2013). We are still in the process of evaluating the data (39 grade 9 classes) because it turned out that the variation among the teaching styles was much bigger than initially expected. Every participating teacher had to teach a thirteen-unit lesson with the topic being Pythagoras’ Theorem, beginning with an introduction, including a proof of the theorem, intra-mathematical applications, word problems, and finally modeling problems on the level of grade nine. In two of the three conditions that we had, there were diagnostic sheets for formative assessment, implemented in three of the lessons. The third and last diagnostic sheet contains the modeling task “Cable car,” which fits into this topic area (Figure 5).

This task pictured was positioned on the left of the sheet, together with the student’s solution (not pictured). The right of the sheet was reserved for feedback from the teacher (not pictured). There are several ways of solving the task. One does not need the Pythagorean Theorem, but because the task is given

within the unit on this theorem, nearly all of the students used it. One of the most important aspects of feedback is not only that weaknesses should be reported but also strengths, which are often omitted. Thus, the first part of the feedback is: “What are you already quite good at?” Subsequent is feedback such as “You can still improve in the following aspects that I saw in your solution if you follow my hints,” followed by some corresponding hints. One of the research questions asked what the effects of the teachers’ hints were. There was a parallel version of this “Cable car” task in the post-test of the study, so we could compare how the students solved this task in the diagnostic sheet and how they solved it on the final test. The hope was, of course, that everybody would be able to solve it on the final test because the diagnostic sheet revealed all the problems and the students received individual written feedback. What were the effects of this written feedback? We have identified certain patterns. The feedback was mostly successful if the teacher gave both a strategic hint on the meta-level and a hint referring concretely to the task. Here is an example: A student’s solution was nearly correct (apart from an inappropriate preciseness in the digits) but he forgot to double his result because there are two ropes in the cable car, one up and one down (as can be seen in the photo). The teacher reported back to the student: “You made a mistake. You didn’t consider the two rope,” and he gave the meta-hint: “Look closely at the picture!” The same student provided a correct solution on the final test. Also noteworthy in this case, the teacher’s feedback about rounding off was successful. Therefore, it is a combination of reference to the task and meta-level that helped. In other cases, when there was either only a meta-hint without reference to the task or only a concrete reference to the task without a meta-hint, the feedback was often not successful. An example of such a case: A student who made the same mistake of not doubling, and the teacher only gave the meta-hint: “Imagine the situation.” This was not successful, and the student made precise

The rope of the cable car *Ristis* has to be replaced. 1 m of the rope costs 8 €. How much does a new rope cost approximately?

Write down your solution process.




Name:	Cable car „Ristis“	
Station 1:	1600 meter above sea level	
Station 2:	1867 m above sea level	
Horizontal difference:	869 m	
Weight capacity:	132 · 3 persons	
Haul capacity:	1200 persons per hour	
Speed:	1,5 m/s	
Time of travel:	10 min	

Figure 5.

the same mistake on the final test. So strategic interventions are important but not sufficient, and there must also be a link to the concrete task.

I would like to close this part of the paper by emphasizing that a lot of competencies are necessary on the teacher's side for teaching modeling (e.g., Doerr, 2007; Kaiser, Schwarz & Tiedemann, 2010). First of all, all the competencies that the students ought to achieve have to be achieved by the teacher himself/herself. Important parts of the pedagogical content knowledge of teachers comprise a theoretical dimension such as knowledge of aims for modeling and modeling cycles, a task dimension with analysis and construction of modeling tasks, an instructional dimension, and a diagnostic dimension (see Borromeo Ferri & Blum, 2010).

A Teaching Unit for Modeling in the Ninth Grade

I would like to present a teaching unit for modeling in the ninth grade on the Pythagorean Theorem, but one that also included other modeling tasks, in particular, tasks related to the study of linear functions. In the DISUM project (see Schukajlow, Leiss, Pekrun, Blum, Müller, & Messner, 2012) we have developed a so-called operative-strategic way of teaching that tries to incorporate several of these quality teaching aspects discussed in the third section. The guiding principles were that:

- The teacher's guidance should aim at students' active and independent solutions;
- There was a systematic change between independent work in groups (individual, pairs, whole group, then individual again) and whole-class activities (for students' presentations and retrospective reflections); and
- Students' work and teachers' coaching should be based on the four-step "Solution Plan" (see section 3).

We confronted this teaching with what we call *directive* teaching. The guiding principles here were:

- The development of common solution patterns for the whole class; and
- A systematic change between whole-class teaching oriented towards the "average student" and students' individual work on exercises.

This kind of teaching is what, according to classroom observations both in Germany and in the United States, characterizes about ninety percent of everyday teaching. In some sense this is the most demanding way of teaching because the teacher has always to be in control of everything, yet will never know what the majority of the students really do. We modeled these two ways of teaching in an ideal-typical way and tried to implement them as optimized teaching styles, with teachers especially trained for that purpose. All were experienced teachers from a reform project in Germany called SINUS. Thus, it was not simply good teaching versus bad teaching, but rather a study concerning two optimized ways of teaching. The unit comprised ten lessons with a pre- and post-test, approximately ten lessons with modeling training, and tasks accessible to grade nine students, which included the "Filling up" task among others. The solution plan was introduced in the third lesson, and in the final two lessons there was the individual practicing of modeling.

What were the results? In the first phase of the project, when students had no solution plan, there were no differences between the two groups with regard to their progress in solving intra-mathematical technical tasks. However, the really interesting result is that only the operative-strategic group made significant progress in modeling tasks. So, after ten lessons of training in modeling with the directive teaching style, it had no effect; although the teachers tried to be as effective as possible, they were only effective in the technical part of it, i.e., not in modeling sub-competencies, such as proposing assumptions or interpreting mathematical results. In the second phase of the project, the students had the solution plan, but now the students in the operative-strategic teaching made significantly more progress in modeling tasks (by one standard deviation).

In conclusion, the above mentioned seems to be a promising approach. What we have not yet done (because the DISUM project was carried out prior to the CO²CA project) is to implement the appropriate use of feedback into the operative-strategic teaching style. We assume that, if implemented correctly, this will lead to even more promising results.

References

- Aebli, H. (1985). *Zwölf Grundformen des Lehrens*. Stuttgart: Klett-Cotta.
- Besser, M., Blum, W. & Klimczak, M. (2013). Formative Assessment in Everyday Teaching of Mathematical Modelling: Implementation of Written and Oral Feedback to Competency-Oriented Tasks. In: Stillman, G. et al. (Eds.), *Teaching Mathematical Modelling: Connecting to Research and Practice*. New York, Springer, 469–478.
- Blum, W. (2011). Can Modelling Be Taught and Learnt? Some Answers from Empirical Research. In: Kaiser, G. et al. (Eds.), *Trends in Teaching and Learning of Mathematical Modelling (ICTMA 14)*. Dordrecht: Springer, 15–30.
- Blum, W., Galbraith, P., Henn, H.-W. & Niss, M. (Eds.). (2007). *Modelling and Applications in Mathematics Education*. New York: Springer.
- Blum, W. & Leiß, D. (2006). “Filling up” – The problem of independence-preserving teacher interventions in lessons with demanding modelling tasks. In: Bosch, M. (Ed.), *CERME-4 – Proceedings of the Fourth Conference of the European Society for Research in Mathematics Education*. Guixol.
- Blum, W. & Leiß, D. (2007). How do students’ and teachers deal with modelling problems? In: Haines, C. et al. (Eds.), *Mathematical Modelling: Education, Engineering and Economics*. Chichester: Horwood, 222–231.
- Borromeo Ferri, R. (2007). Modelling problems from a cognitive perspective. In: Haines, C. et al. (Eds.), *Mathematical Modelling: Education, Engineering and Economics*. Chichester: Horwood, 260–270.
- Borromeo Ferri, R. & Blum, W. (2010). Mathematical Modelling in Teacher Education – Experiences from a Modelling Seminar. In: Durand-Guerrier, V., Soury-Lavergne, S. & Arzarello, F. (Eds.), *CERME-6 – Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education*. INRP, Lyon 2010, 2046–2055.
- Burkhardt, H. & Pollak, H.O. (2006). Modelling in mathematics classrooms: reflections on past developments and the future. In: *Zentralblatt für Didaktik der Mathematik*, 38(2), 178–195.
- Doerr, H. (2007). What knowledge do teachers need for teaching mathematics through applications and modelling? In: Blum, W. et al. (Eds.), *Modelling and Applications in Mathematics Education*. New York: Springer, 69–78.
- Galbraith, P./ Stillman, G. (2006). A framework for identifying student blockages during transitions in the modelling process. In: *Zentralblatt für Didaktik der Mathematik*, 38(2), 143–162.
- Hattie, J. & Timperley, H. (2007). The power of feedback. *Review of Educational Research*, 77(1), 81–112.
- Kaiser, G. (2007). Modelling and modelling competencies in school. In: Haines, C. et al. (Eds.), *Mathematical Modelling: Education, Engineering and Economics*. Chichester: Horwood, 110–119.
- Kaiser, G., Schwarz, B. & Tiedemann, S. (2010). Future Teachers’ Professional Knowledge on Modeling. In: Lesh, R., Galbraith, P.L., Haines, C.R. & Hurford, A. (Eds.): *Modeling Students’ Mathematical Modeling Competencies. ICTMA 13*. New York: Springer, 433–444.
- Maaß, K. (2006). What are modelling competencies? In: *Zentralblatt für Didaktik der Mathematik*, 38(2), 113–142.
- Niss, M. & Højgaard Jensen, T. (Eds.). (2011). *Competencies and Mathematical Learning*. Roskilde University.
- Pollak, H.O. (1969): How can we teach applications of mathematics? In: *Educational Studies in Mathematics* 2, 393–404.
- Pollak, H.O. (1979). The Interaction between Mathematics and Other School Subjects. In: UNESCO (Ed.), *New Trends in Mathematics Teaching IV*. Paris, 232–248.
- Schoenfeld, A.H. (1994). *Mathematical Thinking and Problem Solving*. Hillsdale: Erlbaum.
- Schukajlow, S., Leiss, D., Pekrun, R., Blum, W., Müller, M. & Messner, R. (2012). Teaching methods for modelling problems and students’ task-specific enjoyment, value, interest and self-efficacy expectations. In: *Educational Studies in Mathematics* 79(2), 215–237.
- Stillman, G. (2011). Applying Metacognitive Knowledge and Strategies in Applications and Modelling Tasks at Secondary School. In: Kaiser, G. et al. (Eds.). *Trends in Teaching and Learning of Mathematical Modelling (ICTMA 14)*. Dordrecht: Springer, 165–180.
- Treilibs, V., Burkhardt, H. & Low, B. (1980). *Formulation processes in mathematical modelling*. Nottingham: Shell Centre for Mathematical Education.
- Verschaffel, L., Greer, B. & DeCorte, E. (2000). *Making Sense of Word Problems*. Lisse: Swets&Zeitlinger.