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Promoting Equitable Practices in Mathematics Education

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Nurturing the Generation and Exploration of Mathematical Conjectures with Preservice Teachers: An Example with a Perimeters Task

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ABSTRACT Generating mathematical conjectures is an important mathematical habit of mind for preservice teachers to develop so that they can, in turn, help their students to develop this skill. In this paper we present a classroom episode in which preservice mathematics teachers experience conjecturing in the context of a rich, open-ended task. They also reflect, throughout the task, on how they might nurture the generation and exploration of mathematical conjectures with their own students.

KEYWORDS *mathematical conjectures, preservice teachers*

Introduction

Doing mathematics involves discovery. Conjecture—that is, informed guessing—is a major pathway to discovery (NCTM, 2000, p. 57).

The National Council of Teachers of Mathematics (NCTM, 2000) defines conjecturing, an important aspect of learning about proof, as informed guessing and highlights it as one of the major components of mathematical reasoning. Furthermore, the NCTM states that "reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts" (p. 56). In a review of the literature on mathematics teachers' conceptions of proof, Ko (2016) reported evidence of large-scale issues with teacher knowledge regarding both their own understanding of, and their work with students on, mathematical proof. Therefore, it is important to think about how we work on conjecturing with preservice mathematics teachers throughout their period of induction so that they, in turn, can support their students in the conjecturing process.

One approach to this is by modeling conjecturing experiences for preservice teachers thereby placing them in the position of learners as they explore a task without knowing the result. In attempting to do this, we face the challenge of finding suitable tasks and scaffolding them in a way that maximizes opportunities to learn. This means that we should aim to scaffold enough to make a task tractable for students but not so much that they are simply following step-by-step instructions without opportunities to make mathematical decisions and discoveries for themselves. An important consideration is how much time a teacher should allow for the early stages of conjecturing, especially based on a single case—how many cases are enough? How can we help students relate those multiple cases? Do those new cases cause reformulation of the conjecture? How do we keep track of the evolution of a conjecture? In this paper we present an example, through a classroom episode, of one encounter with these considerations and describe how the classroom episode provided preservice teachers both with the experience of developing conjectures and with an opportunity to reflect on the process from a teacher's perspective.

Relationship to Literature

Astawa et al. (2018) state that "constructing mathematical conjecture involves a lot of [complex] processes of cognition" (p. 17) and emphasize the problem-solving aspect of conjecturing. Ponte et al. (1998) list the main three stages of conjecturing as "(i) proposing questions and establishing conjectures, (ii) testing and refining the conjectures, and (iii) arguing and proving the conjectures," (p. 4). Others, for example, Cañadas et al. (2007) and Morseli (2006) also highlight its multi-stage nature and, while stages suggest a hierarchy, we argue that conjecturing is cyclic. After a conjecture is formulated, it is often reformulated (perhaps multiple times) and refined until validated or proven. Moreover, many researchers (e.g. Astawa et al., 2018; NCTM, 2000) emphasize the intertwined relationship between exploration, discovery, and conjecturing.

Cañadas et al. (2007) identify five distinct types of conjecturing: Type 1: Empirical Induction from a Finite Number of Discrete Cases, Type 2: Empirical Induction from Dynamic Cases, Type 3: Analogy (to something already known), Type 4: Abduction (conjecturing on the basis of a single event), and Type 5: Perceptually Based Conjecturing (made from a visual representation of a problem). The first of these is the most common type in school mathematics with Type 2 becoming more common as dynamic geometry environments gain traction.

In this paper, we will share a classroom episode in which preservice teachers generate conjectures and attempt to validate those conjectures using a Type 4 and then a Type 1 (Cañadas et al., 2007) approach. We follow Driscoll's (2010) suggestion and track the formulation and reformulation of conjectures since, "Studies show that opportunities to explore and conjecture...followed by challenges to explain why conjectures seem true, hold promise for helping students become more proficient at constructing deductive proofs" (Driscoll, 2010, p. 25). The episode provides an example where an appropriate amount of scaffolding provided mathematical space for preservice teachers to work as mathematicians in their conjecturing. Finally, the episode describes how reflection on the process from a teacher perspective can be built into the enactment of the task in the classroom.

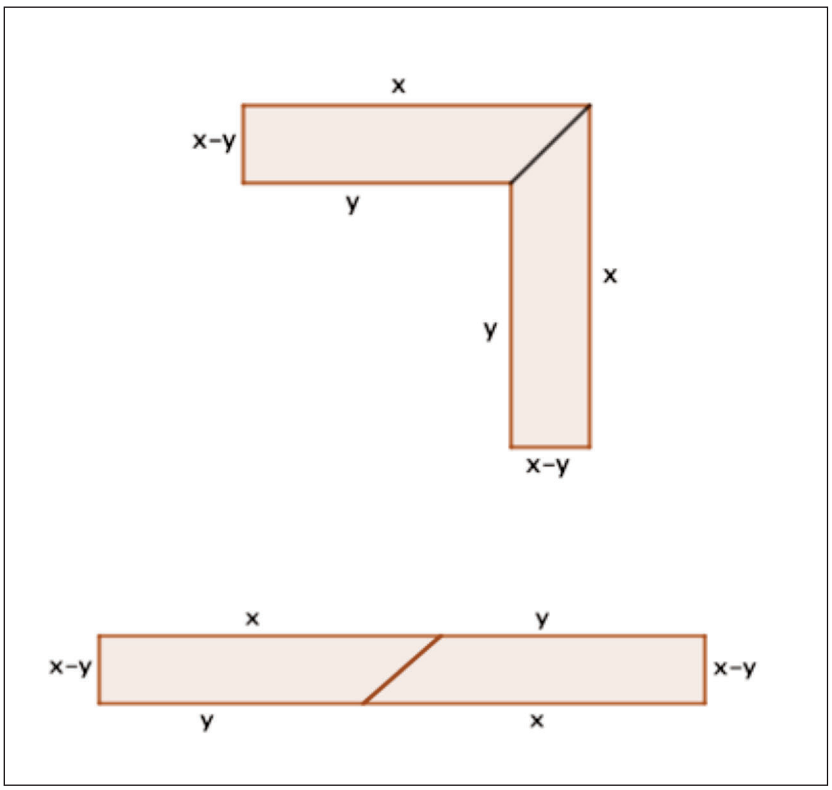
Classroom Episode

In this classroom episode, preservice teachers began working on a task that was designed to provide them with an example of visualizing algebraic relationships. The original task was modified with differences leading to a rich, extended exploration in which preservice teachers could develop conjectures without aiming to establish a known result, i.e., they were exploring as mathematicians in a true sense. The content of the task, perimeters of shapes, was especially useful because, while the content itself was accessible, the mathematical results that would arise were not known in advance.

Brumbaugh and Rock (2012) include a mathematical activity designed to provide a visualization of the difference of two squares identity, $x^2 - y^2 = (x - y)(x + y)$. One of the authors has used this activity regularly in his class: preservice teachers are led through the task starting with a large square and cutting out a smaller square to get an L shape (see Figure 1 below). The larger square has area x^2 and smaller square has y^2 so the area of the L shape is $x^2 - y^2$. But the L shape can also be cut along the diagonal and the resulting two pieces arranged to make a rectangle of dimensions $x + y$ by $x - y$ (see lower portion of Figure 1). Hence, we have a visual demonstration of the difference of two squares identity, $x^2 - y^2 = (x - y)(x + y)$.

Figure 1

Visual Demonstration of Difference of Two Squares Identity



In past iterations of the course, preservice teachers were given the task and led through all steps, including cutting the L, rearranging into a rectangle, and labeling. However, on this occasion, the task was paused when students had generated the L shape on the left (seen in Figure 1) and, rather than being led through the subsequent steps, were simply asked to explore further. A number of the preservice teachers went to the result in the original task, i.e., cutting the L, calculating areas, and finding the difference of two squares result. As we describe in the classroom episode below, sometimes scaffolding less is beneficial and can lead to unexpected (and genuinely interesting) results.

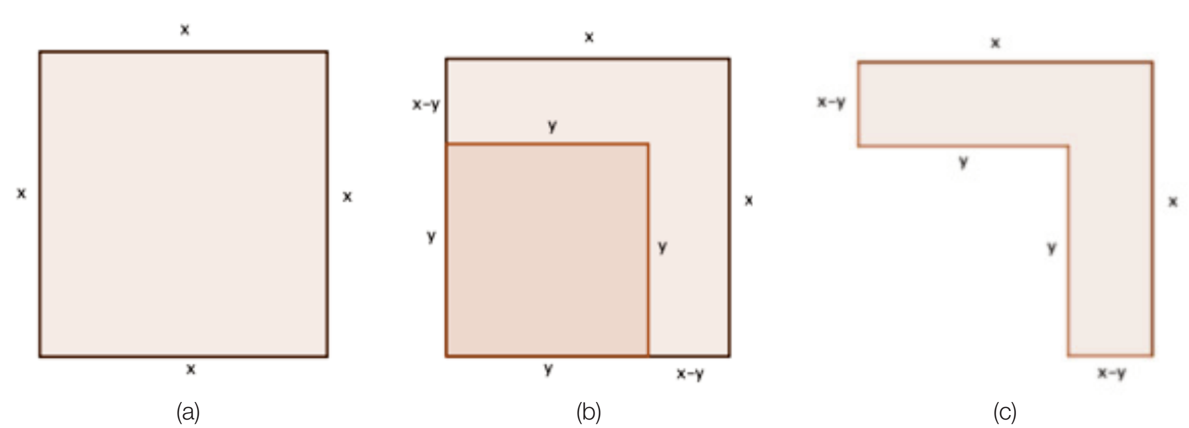
In the original task the focus is on calculating areas. On this occasion, not having been directed to examine the areas of the shapes, other preservice teachers took the path of examining perimeters of the shapes. As can be seen in the sample work in Figure 2, they quickly concluded that the L-shape and the large square had the same perimeter. Thus, per Cañadas et al., (2007) for the purposes of making conjectures, we had just one case to consider (Type 4: Abduction).

As is often the case in mathematics, a calculation leads to the question "Did we actually need to do the work of that calculation or could we have reasoned our way to the result?" Indeed, in this case, it was argued that 'y' parts of the original perimeter were moved to a "different place" but still contributed to the perimeter of the resultant shape, so the work of the calculation was not necessary. Moreover, this particular result prompted the realization by the preservice teachers that, of course, the length of the smaller square is arbitrary and that any square cut from a corner of the larger square would result in a shape with the same perimeter. Thus, per Type 4 of Cañadas et al. (2007), a single example resulted in a conjecture and a general argument without the need to generate more examples. In addition, the process prompted several different directions for the generation of more cases as preservice teachers looked to uncover patterns, thus adopting a Type 1 approach per Cañadas et al, to wit, "a conjecture can be made based on the observation of a finite number of discrete cases, in which a consistent pattern is observed" (p. 58). Of course, it was not clear which aspect of the task should be explored so that further cases could be generated. In the moment in that class, our initial observation and some brainstorming prompted three avenues for further exploration, namely: (a) Could we cut any regular polygon from a large regular polygon of the same kind and preserve the perimeter? (b) Did it matter from where the small square was cut, i.e., did it have to be from the corner? and, (c) Did it matter that the original shape was a square, i.e., could we cut out a different shape?

Here, we highlight for preservice teachers the importance of not over scaffolding, specifically, not providing too much explicit direction. The previous, highly scaffolded, area task, was designed to direct students towards a particular result (a visual demonstration of the difference of two squares identity). Allowing some time for undirected exploration of the task allowed preservice teachers to find new avenues of exploration. The instructor's crucial role during such times of exploration is to

Figure 2

Perimeters of Original Square (a) and L-Frame (c)



Note. Original square perimeter (a): $x + x + x + x = 4x$. Frame perimeter (c): $y + y + (x - y) + x + x + (x - y) = 4x$.

monitor groups and see which, if any, pose interesting questions warranting further exploration. The instructor also must be mindful of bringing the group back together for the kind of brainstorming that produced the three questions above. This has multiple positive effects: (i) it allows the whole class to decide on a common focus for further exploration; (ii) it allows those groups who have started productive lines of inquiry to share their suggestions; and (iii) it provides a productive line of work for those students who struggled in the undirected exploration time.

This is also a demonstration of the power of open-ended exploration: an observation might be followed with multiple conjectures to be tested. Interestingly, in this particular case, the exploration of the perimeters, rather than the areas, meant that the task had become a new one for the instructor, who was then faced with the decision of taking time to explore avenues which may or may not be productive. So, what should an instructor do at this moment? We all know that instructional time is precious, and too few teachers afford students the opportunity to follow their conjectures. Moreover, an instructor might feel uneasy to continue with further exploration, especially not knowing exactly how these avenues would play out. Much preservice mathematics teacher education is concerned with modeling good practice since, as in any classroom, instructor moves give tacit messages to students about what matters in a classroom. Tasks such as the one described in this classroom episode show how student reasoning can be at the center of the class and that time spent exploring different avenues of inquiry is time well spent. Since this was a class of preservice teachers, this decision also became a point of discussion in the class. In the discussion, important points were raised about the Mathematical Knowledge for Teaching (MKT) (Hill et al., 2005) that teachers need in order both to guide the brainstorming for ideas and to be able to judge the productive potential of the ideas. In this case, with some ideas rejected, the instructor decided that each of the three ideas had potential and that different groups would pursue different conjectures.

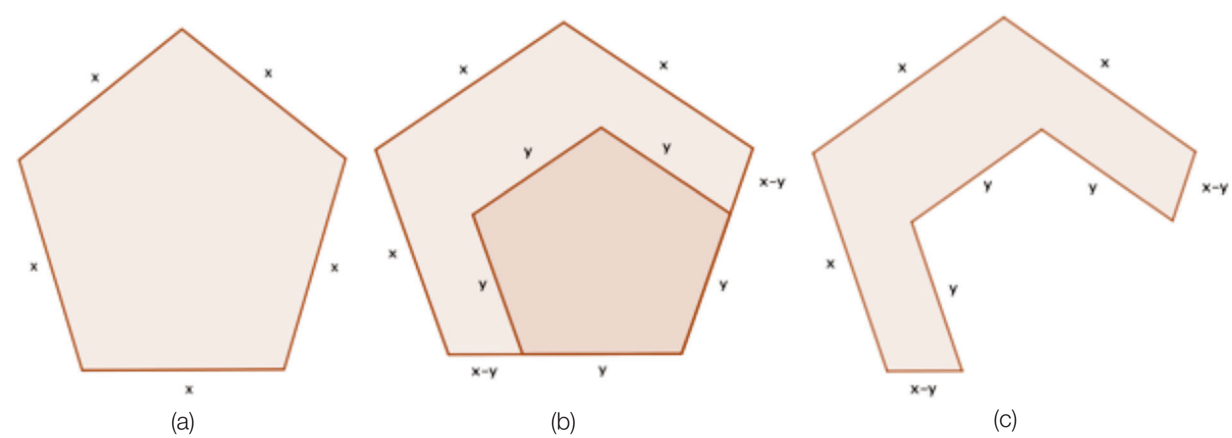
Other regular shapes:

One group of preservice teachers took a Type I (Cañadas et al., 2007) approach as in "observing cases, organizing cases, searching for and predicting patterns, and formulating a conjecture" (p. 63). They began to explore cutting small regular shapes from the corner of a larger version of the same shape to generate several cases and look for a pattern. Figures 3 and 4 highlight work with a regular pentagon and regular hexagon, respectively.

Looking at the regular pentagon and hexagon constructions, it can be seen that, similarly to the case of the square, parts of the original perimeter are "moved"; although in the new bases seen in Figures 3 and 4, new pieces of perimeter appear. Studying these constructions, and considering the square from earlier, a pattern emerged, particularly in regard to how many new pieces appear—no new pieces for the square, one new piece for the pentagon, two new pieces for the hexagon—leading to the formulation of a conjecture: *"When a small regular n-gon of side y is cut from the corner of a large version of the*

Figure 3

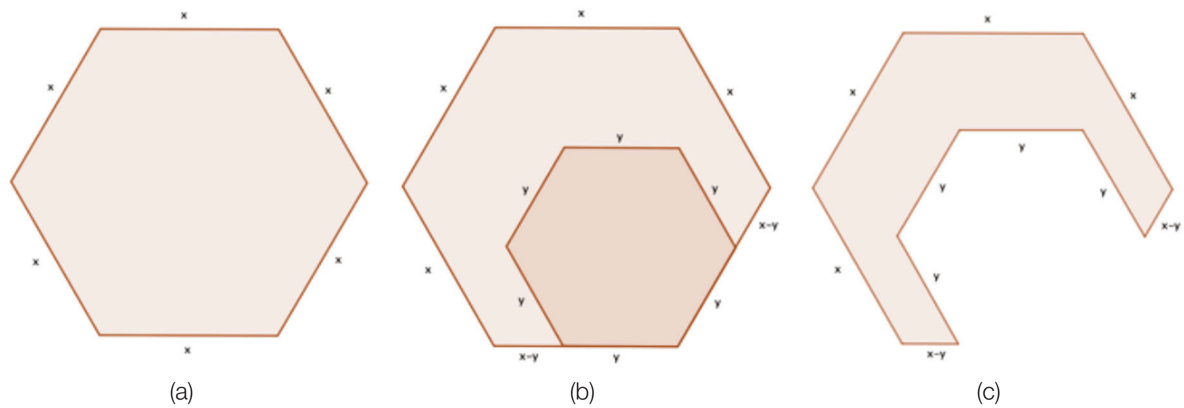
Perimeters of a Regular Pentagon (a) and Frame (c)



Note. Original pentagon perimeter (a): $x + x + x + x + x = 5x$.
Frame perimeter (c): $x + x + x + x + x - y + y + y + y + x - y = 5x + y$.

Figure 4

Perimeters of a Regular Hexagon (a) and Frame (c)



Note. Original pentagon perimeter (a): $x + x + x + x + x + x = 6x$.
Frame perimeter (c): $x + x + x + x + x - y + y + y + y + y + x - y = 6x + 2y$.

regular n-gon of side x, the resulting perimeter is increased by (n – 4)y, i.e., the perimeter increases from nx to nx + (n – 4)y." The result worked for the original shape of the square where the perimeter stayed the same. In that case, $n = 4$ and so the new perimeter is $4x + (4 - 4)y = 4x$.

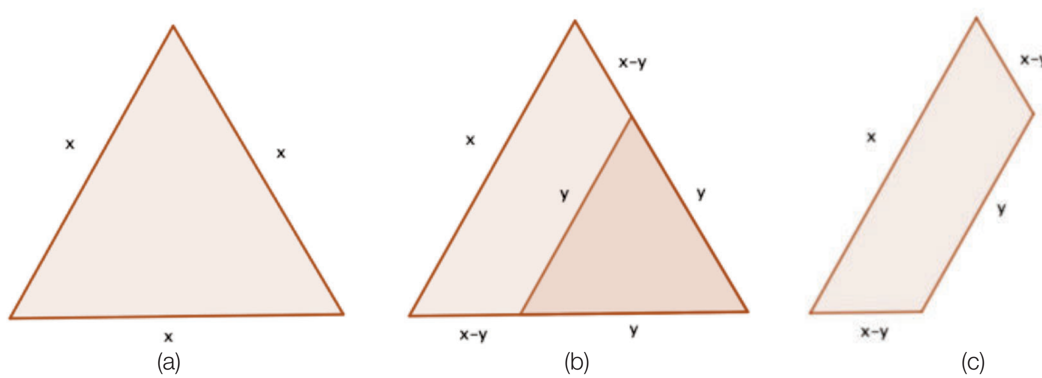
The preservice teachers agreed that the result was reasonable since when a regular shape is cut, two edges of the smaller polygon are "lost" from the original perimeter but are "moved" to the interior. That accounts for 4 edges and the remaining edges ($n - 4$) are added to the perimeter of the new shape. One preservice teacher pointed out that we had not tested the conjecture for triangles and that we would run into trouble since a triangle had fewer than 4 sides and thus $n - 4$ would be negative. However, the diagram provided in Figure 5 confirmed that the conjecture held, although this is the only case where the perimeter decreases. The preservice teachers were satisfied that the result held in all cases, although they were not entirely clear on what a rigorous proof of the result might look like.

Location of the cut:

Other preservice teachers explored the question of the location of the cut by cutting a square from a place on one side rather than from a corner. As Figure 6 illustrates, they soon discovered that the length, 'y', of the smaller square that lay on the perimeter is "moved up," with two new 'y' lengths added, yielding a new perimeter of $4x + 2y$.

Figure 5

Testing Conjecture for Equilateral Triangles



Note. Original equilateral triangle perimeter (a): $x + x + x = 3x$. Frame perimeter (c): $x + x - y + y + x - y = 3x - y$
[Also $nx + (n - 4)y$ when $n = 3$ is $3x + (3 - 4)y = 3x + (-1)y = 3x - y$.]

Figure 6

Cutting a Square from a Non-Corner

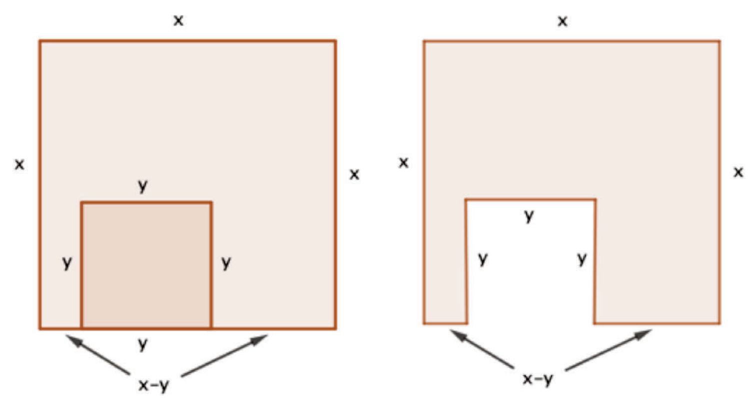
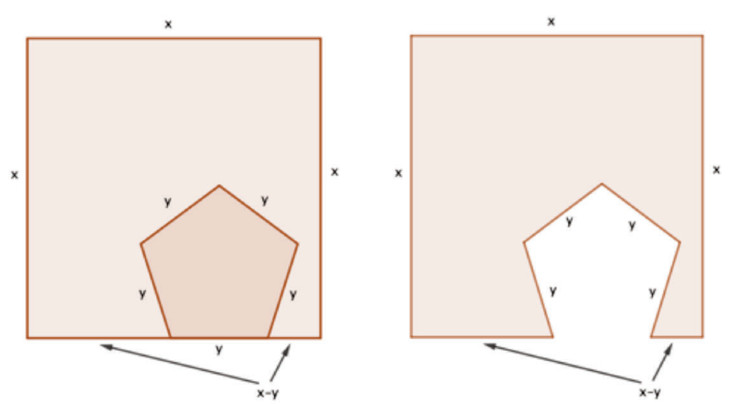


Figure 7

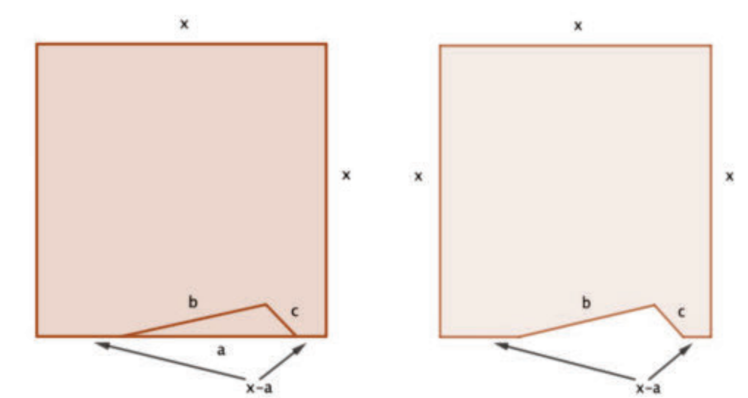
Cutting a Regular Pentagon from a Non-Corner



Note. Perimeter of original square (left): $x + x + x + x = 4x$. Frame Perimeter (right): $x + x + x + x - y + y + y + y + y + x - y = 4x + 3y$.

Figure 8

Cutting a Triangle from a Non-Corner



This prompted a new conjecture: *"If a small square is cut from an edge (rather than a corner) of a large square, the perimeter is increased by exactly half the perimeter of the small square."* Next, students sought to generalize the conjecture by cutting a regular pentagon from the original square, as shown in Figure 7. As groups shared their work, this was identified as a slight variation of an earlier conjecture with only one 'y' "lost," yielding a perimeter of $nx + (n - 2)y$. This was confirmed for the case of the square. When $n = 4$, the perimeter is $4x + (4 - 2)y = 4x - 2y$.

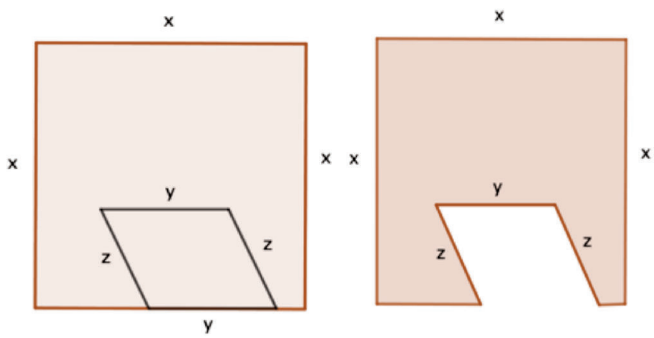
Considering other shapes:

Perhaps the most interesting moment in the activity happened when a pair of preservice teachers exclaimed, "This is an application of the triangle inequality theorem!" The students recalled that "the triangle inequality theorem says that the sum of those two interior sides must be greater than the third side so the perimeter must have increased." Still considering the original question of whether the perimeter is preserved when a shape is cut from it, they declared: "If you cut a triangle from a side of the square there will be one side of the triangle missing from the edge and two new sides on the interior." This is illustrated in Figure 8.

Preservice teachers explored a wide variety of different shapes that could be cut from a corner and from an edge. They quickly asserted and demonstrated that cutting a rectangle from the square leads to the same two conjectures as before. This is illustrated in Figure 9, with the modification that, for the second conjecture, the extra perimeter will come from the two sides of the rectangle—neither of which overlapped with the edge of the original square. Noting that you cannot cut a parallelogram from the corner, the second theorem held for a parallelogram cut from an edge, as shown in Figure 9. As preservice teachers cut other shapes, similar patterns of

Figure 9

Cutting a Parallelogram from a Non-Corner



Note. Perimeter of original square (left): $x + x + x + x = 4x$.
Frame perimeter (right): $x + x + x + x - y + y + z + z = 4x + 2z$

"lost" sides being "regained" in the interior, and other side lengths emerged.

The three lines of inquiry—regular shapes, the location of the cut, non-regular shapes—all led to interesting mathematical results aided by the instructor taking an active role in circulating among the groups and monitoring progress. At the conclusion, a discussion of how the activity was managed by the instructor and the main pedagogical choices made (not over scaffolding, time for free exploration, consolidation of ideas from free exploration, focusing the groups on agreed lines of inquiry) gave the preservice teachers an opportunity to reflect on the role they could take with their own students going forward.

Concluding Thoughts

Having the opportunity to explore mathematical tasks and to develop conjectures is an important part of developing mathematical skills and developing maturity as a mathematician. As Ko (2010) has reported in a review of the literature, preservice teachers would benefit from continuous and ongoing opportunities to engage in such activity. Preservice teachers, therefore, need tasks which provide useful avenues for exploration but that are not overdetermined with too much scaffolding so that they can both develop skills in exploration for themselves and reflect on how they would engage their own students in this kind of work. The task discussed in the classroom episode above provides an example of such a task. The open-ended nature of the task gave preservice teachers several avenues of exploration, engaged the preservice teachers in several types of conjecturing (Cañadas et al., 2007) and led to some unexpected results such as an application of the triangle inequality theorem. The task also provided the instructor an opportunity to demonstrate how to orchestrate various groups of students exploring various conjectures. This in turn showed preservice teachers how to help their future students value each others' conjectures and recognize how they are interrelated, inform one another, and can lead to a deeper understanding of the concepts at hand. Finally, the setting of a preservice teacher classroom allowed for "meta" discussion about how the task was being conducted by the instructor and the pedagogical choices that were being made.

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