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Inverted Tasks and Bracketed Tasks in Mathematical Problem Posing

Benjamin Dickman
The Hewitt School

ABSTRACT We present in this paper a pair of approaches to support mathematics educators and learners in formulating original tasks. In particular, we facilitate the posing of rich mathematical problems by using two novel methods that were created by a mathematics department at a K-12 school in the United States, and further developed alongside our students as well as a wider professional learning team of master teachers. We situate our work within the broader literature on mathematical problem posing and describe our strategies by including examples of their use in generating problems and by providing examples of authentic student-assigned tasks that were created with our approaches.

KEYWORDS *problem posing, assessment, evaluation, task design, inverted task, bracketed task*

Introduction

The literature on mathematical problem posing can be traced back at least to Polya's (1945) "How to Solve It," for which many of the heuristics around problem solving involve the asking of questions: *What is a related problem? What is a simpler problem? How can the problem be generalized?* The observation that problem solving involves posing, or reformulating, problems dates back at least to another work of the same year as remarked by Kilpatrick in Schoenfeld (1987, p. 125):

> Wherever the problem comes from, the problem solver is always obliged to reformulate it. In fact, as Duncker (1945) pointed out, one can think of problem solving itself as consisting of successive reformulations of an initial problem.

Other classical works on problem posing include Silver's (1994) article that connects problem posing with creativity, and Brown and Walter's (1983) treatise, "The Art of Problem Posing," in which an explicit model is described for modifying a given scenario (e.g., the graph of $y = x^2 + 6x + 9$) to create original problems: attribute listing, which involves writing out the various traits and characteristics of the initial setup (e.g., this quadratic function has one real root; the y-intercept is at $y = 9$); what-if-not-ing, which involves the new scenarios produced by asking about the changes effected if an attribute were assumed not to hold (e.g., What if the quadratic function had two real roots? What if the y-intercept were not 9?); and cycling, which involves combining multiple variations on a scenario to create a novel problem (e.g., What is a quadratic function that has two distinct real roots and a y-intercept at $y = 8$?).

We build on this earlier work with a pair of approaches to problem posing: the first approach, which we refer to as *inverted problem posing,* involves a new perspective on the relation between inputs, outputs, and methods/algorithms in mathematics courses; the second approach, which we refer to as *bracketed problem posing*, involves an approach to encouraging problem posing among teachers and learners of mathematics, and which we have incorporated into our own assessments. In doing so, we aim to answer the call of Kilpatrick that "problem formulating should be viewed not only as a goal of instruction, but also as a means of instruction. The experience of discovering and creating one's own mathematics problems ought to be part of every student's education" (1987, p. 123).

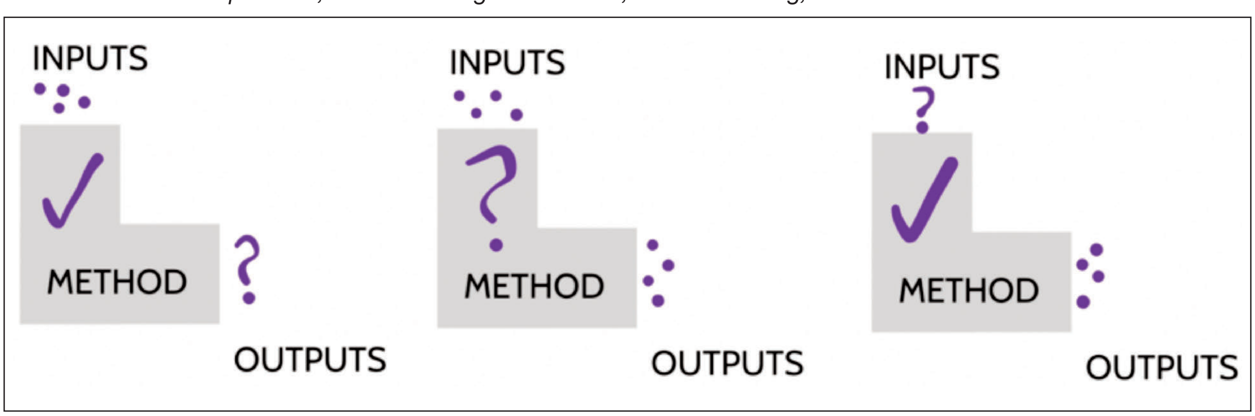
Inverted Problem Posing

We begin (see Figure 1) by characterizing three different approaches to mathematical problems in terms of their input (what is given), method of solution (technique or algorithm), and output (result of running the input through the method). For example, the input could be "4 and 6," the algorithm could be one to find the least common multiple (LCM), and the output would, therefore, be LCM(4,6) = 12. In traditional approaches to teaching mathematics, instructors might teach a method first (e.g., related to the prime factors) and then have students practice by giving them lots of inputs (here, lots of number pairs); so, the inputs are given, the method is known, and various outputs are found through a sequence of exercises. In a problem solving approach, instructors may set up an exploration in which the inputs are given or can be chosen, and students can find specific outputs. The challenge for students becomes the creation of algorithms or methods, scaffolded as necessary, to generalize their solution technique; so, individual inputs and outputs are available, but the method is to be found through investigation. For example, students may compute the LCM of many pairs of numbers and look for patterns to develop an algorithm that yields the LCM in general. We use *inverted problem posing* to refer to a different model: once students have understood a method, we provide sample outputs and ask for the possible inputs. For example, rather than providing natural numbers a and b for which LCM(a,b) = 12, we ask: Given that LCM(a,b) = 12, what could be the values of a and b? One consequence of this approach to problem posing is that the inversion may produce multiple inputs; in our example here, we could have, e.g., $a = 4$ and $b = 6$, or $a = 2$ and $b = 12$. Further employing Brown and Walter's (1983) scheme (what if there were not multiple answers?) can allow one to formulate problems with unique answers: one workaround is simply to ask for the set of all possible solutions; another approach is to impose additional constraints, e.g., for what a and b satisfying LCM(a,b) = 12 do we minimize $a + b$?

One of the advantages of inverted problem posing is that instructors can deploy this strategy whenever a problem has been solved, or an attribute has been noticed. For example, we used the Brown and Walter (1983) scheme to phrase our formulation of an earlier problem: What is a quadratic function that has two distinct real roots and a y-intercept at $y = 8$? This required a combination of attribute listing, what-if-not-ing, and cycling. The same question could, instead, be viewed as an inverted problem from the routine question, "What are the two roots and y-intercept of the quadratic function $y = x^2 + 6x + 8$?" Moreover, we can notice and incorporate additional features of this parabolic curve to pose richer problems. For example, the x-intercepts in this graph are at $x = -2$ and $x = -4$; so, they have a distance of 2 between them as measured along the x-axis. Given this additional observation, we may now consider the inverted problem that asks, "What is a quadratic function that has x-intercepts that are 2 apart, and a y-intercept at $y = 8$?" We can further enrich the problem by asking for all the quadratic functions that satisfy these criteria. Calling one root p, we have roots p and $2 + p$. Thus, the quadratic function has the form $y = a(x - p)(x - (2 + p))$, which has constant term $ap(2 + p) = 8$, i.e., $a = 8/(p(2 + p))$. An animated graph of this function, for which the parameter p varies between –5 and 5, can be found on Desmos [https://www.desmos.com/calculator/tr8yrpqalh]. In this way, we have taken the common context of quadratic functions and observations around standard features of their graphs, x-intercepts and the y-intercept, and inverted this information so as to pose a non-routine problem with a low-floor (e.g., give one example) and high-ceiling (e.g., parameterize all such functions).

Figure 1

Models of Problem Exploration, from Left to Right: Traditional, Problem Solving, Inverted



Bracketed Problem Posing

The second method of problem posing described here also builds outwards from standard problems. The general idea is captured by the following directions, which have been used across courses ranging from middle school algebra to high school calculus at the author's institution, as well as by additional master teacher fellows who attended a professional development course ["Problem Posing in Algebra Assessments"] co-facilitated by the author through the organization *Math for America*.

> **Directions:** For each of the following problems, please change only the portion in brackets to: (1) create a similar problem; (2) solve your similar problem; and (3) explain briefly how your problem is similar to the original.

The goal with these directions is to scaffold student problem posing by providing a framework within which they can create their own problems and encourage students to attend to structural rather than superficial features of mathematical tasks. The problem we developed in our previous question could be bracketed for students as follows:

> Show that the two x-intercepts of the graph of the quadratic function $y = [1]x^2 + [6]x + 8$ have a distance of 2 between them as measured along the x-axis.

The placement of the brackets is nontrivial and requires teacher expertise in thinking through what the resulting solution set might look like, how difficult the resulting problem is, and the potential for student misunderstanding of what is being asked. In the example above, directly answering the question is only a matter of finding the real roots (e.g., by factoring) and then verifying that they have a difference of 2. Much more challenging is for students to think through how the degree 2 and 1 coefficients can be chosen to yield another quadratic with real roots that differ by 2. In some cases, this may lead students into an exploration that looks more like our high-ceiling parameterization of the previous section; for others, finding a solution might involve playing with the constraints (e.g., observing $y = x^2 - 1$ has real roots that are 2 apart and multiplying through by -8 to get the desired y-intercept but not change the x-intercepts, i.e., using the quadratic $y = -8x^2 + 8$ as a solution). The considerations of where to incorporate bracketed tasks into a curriculum, how to grade or otherwise use summative assessments (if at all) of student work, and ways of formulating these tasks as a function of teacher goals all fall outside the scope of this brief paper. Instead, we close with a bracketed Algebra 2 examination that was given to students; sample copies of student work are available by request, as are corresponding materials from the professional development course referenced previously.

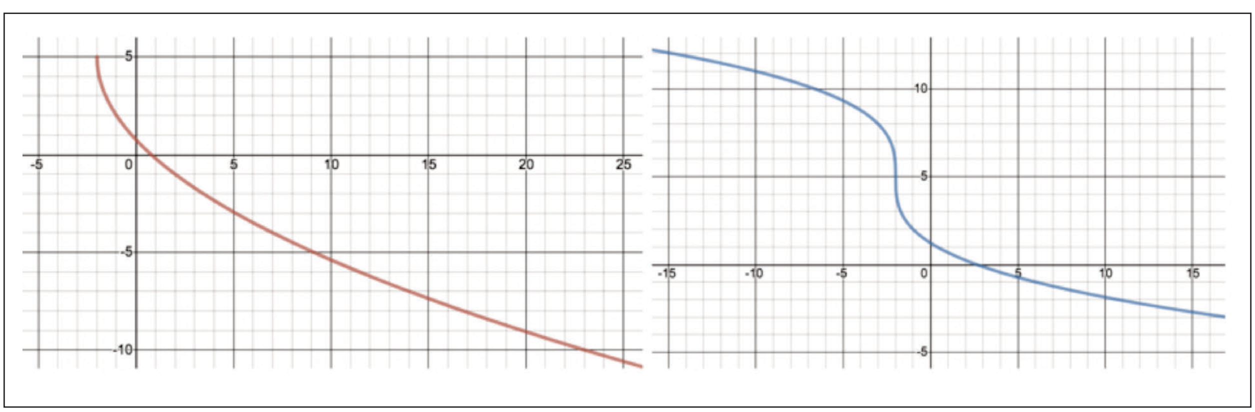
> **PROBLEM 1:** A polynomial with degree 4 has imaginary roots [$2i$ and $3i$]. Give two different possibilities for the polynomial; ensure that your examples do not both have the same end behavior.

> **PROBLEM 2:** Carry out the following polynomial division, by a quadratic expression, by hand in any manner that you wish, and verify that the quotient has a remainder of zero:
> [$(x^6 + 3x^5 - 3x^3 + 6x^2 + 9x + 2) \div (x^2 + 3x + 2)$]

> **PROBLEM 3**: Explain carefully which of the following two questions you would prefer on an in-class test, but you do **not** need to answer either of them.
>
> QUESTION A: Find all the rational roots of [$f(x) = x^5 - x^4 + 2x^3 - 5x^2 + 1$]
>
> QUESTION B: Find all the rational roots of [$g(x) = 2x^3 - 2x + 12$]

Figure 2

Two Bracketed Graphs of Functions Expressible as $y = a(x - h)^{1/n} + k$



PROBLEM 4: Write down radical functions corresponding to the two graphs in Figure 2 and explain carefully how you arrived at your answers. Consider the entire graphs bracketed!

Conclusion

In this paper, we named and briefly discussed two approaches to problem posing: inverted tasks and bracketed tasks. The former involves a reframing of how tasks are created in mathematics classrooms that is situated outside the binary of traditional instruction (tell methods, complete exercises) and certain problem solving alternatives (explore problems, discover methods). The latter involves a scaffolded approach to support student problem posing that encourages learners to focus on structural, rather than superficial, problem features. Our discussion of each is incomplete, and we look forward to developing and refining both—in theory, in practice, and in writing—as our thinking further evolves.

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Author's Note

This article was written for Topic Study Group 17, "Problem posing and solving in mathematics education," which was to be chaired by Teachers College alum Edward A. Silver at ICME-14 in Shanghai, China, during the Summer of 2020. However, the quadrennial International Congress on Mathematical Education has been postponed by at least a year due to the ongoing COVID-19 pandemic. As a bit of background for this 10th anniversary issue of the Journal of Mathematics Education at Teachers College: I arrived as a new doctoral student at Teachers College one decade ago, after spending the better part of the previous two years living in Nanjing, China, which I first called home during a 2008-09 Fulbright Fellowship to research Chinese mathematics teacher education. I am grateful to have had the opportunity, supported initially by the US Department of State, to live in China and form deep friendships during my time abroad. I am grateful to Teachers College for supporting me as an instructor for the graduate course "Teaching Mathematics in Diverse Cultures" while bringing a cohort of doctoral students to Shanghai during a study tour in Summer 2013. I am grateful to my present institution, The Hewitt School, for supporting me in language education studies at Nanjing Normal University in Summer 2017. It is within this context that I must also articulate my deep concern about the xenophobic/Sinophobic and racist rhetoric from United States politicians who were ostensibly elected to lead. My sincere hope is that when we look back in another decade we will see great progress and meaningful structural change, and that the present modes of targeting and othering individuals based on their identities—including, but not limited to, Asian Americans who have faced a recent uptick in hate crimes—will diminish. Yet, hope is sustained by shifting from thought to action, and few actors can be as powerful—and empowering—as educators.

Benjamin Dickman, April 15, 2020