Journal of Mathematics Education at Teachers College

Fall – Winter 2010

A CENTURY OF LEADERSHIP IN MATHEMATICS AND ITS TEACHING

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The *Journal of Mathematics Education at Teachers College* is a publication of the Program in Mathematics and Education at Teachers College Columbia University in the City of New York.

This issue's cover and those of future issues will honor past and current contributors to the Teachers College Program in Mathematics. Photographs are drawn from the Teachers College archives and personal collections.

This issue honors Dr. Alexander P. Karp, an Associate Professor in the Program in Mathematics at Teachers College. A native of St. Petersburg, Russia who is the author of more than one hundred publications including textbooks used throughout Russia, Professor Karp represents Teachers College at meetings and conferences throughout the world as well as through his role as managing editor of the *International Journal for the History of Mathematic Education*.

Former Teachers College Professor and Mathematics Education Chair, Howard Franklin Fehr, was among the most influential mathematics educators of his era. Through his many international contacts, he was the organizer of conferences, projects, and publications including the Congresses of Mathematics Education, a seminal conference on Needed Research in the field, and curriculum initiatives including the Secondary School Mathematics Curriculum Improvement Study.

Aims and Scope

The *JMETC* is a re-creation of an earlier publication by the Teachers College Columbia University Program in Mathematics. As a peer-reviewed, semi-annual journal, it is intended to provide dissemination opportunities for writers of practice-based or research contributions to the general field of mathematics education. Each issue of the *JMETC* will focus upon an educational theme. Themes planned for the 2011 issues are: *Mathematics Curriculum* and *Technology*. *JMETC* readers are educators from pre K-12 through college and university levels, and from many different disciplines and job positions—teachers, principals, superintendents, professors of education, and other leaders in education. Articles to appear in the *JMETC* include research reports, commentaries on practice, historical analyses and responses to issues and recommendations of professional interest.

Manuscript Submission

JMETC seeks conversational manuscripts (2,000-2,500 words in length) that are insightful and helpful to mathematics educators. Articles should contain fresh information, possibly research-based, that gives practical guidance readers can use to improve practice. Examples from classroom experience are encouraged. Articles must not have been accepted for publication elsewhere. To keep the submission and review process as efficient as possible, all manuscripts may be submitted electronically at www.tc.edu/jmetc.

Abstract and keywords. All manuscripts must include an abstract with keywords. Abstracts describing the essence of the manuscript should not exceed 150 words. Authors should select keywords from the menu on the manuscript submission system so that readers can search for the article after it is published. All inquiries and materials should be submitted to Ms. Krystle Hecker at P.O. Box 210, Teachers College Columbia University, 525 W. 120th St., New York, NY 10027 or at JMETC@tc.columbia.edu

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Call for Papers

The "theme" of the spring issue of the *Journal of Mathematics Education at Teachers College* will be *Mathematics Curriculum*. This "call for papers" is an invitation to mathematics education professionals, especially Teachers College students, alumni and friends, to submit articles of approximately 2000-2500 words describing research, experiments, projects, innovations, or practices related to mathematics curriculum. Articles should be submitted to Ms. Krystle Hecker at jmetc@tc.edu by January 1, 2011. The spring issue's guest editor, Nicholas Wasserman, will send contributed articles to editorial panels for "blind review." Reviews will be completed by February 1, 2011, and final drafts of selected papers are to be submitted by March 1, 2011. Publication is expected in mid-April, 2011.

Call for Volunteers

This *Call for Volunteers* is an invitation to mathematics educators with experience in reading/writing professional papers to join the editorial/review panels for the spring 2011 and subsequent issues of *JMETC*. Reviewers are expected to complete assigned reviews no later than 3 weeks from receipt of the blind manuscripts in order to expedite the publication process. Reviewers are responsible for editorial suggestions, fact and citations review, and identification of similar works that may be helpful to contributors whose submissions seem appropriate for publication. Neither authors' nor reviewers' names and affiliations will be shared; however, editors'/reviewers' comments may be sent to contributors of manuscripts to guide further submissions without identifying the editor/reviewer.

If you wish to be considered for review assignments, please request a *Reviewer Information Form*. Return the completed form to Ms. Krystle Hecker at jmetc@tc.edu or Teachers College Columbia University, 525 W 120th St., Box 210, New York, NY 10027.

Looking Ahead

Anticipated themes for future issues are:

Spring 2011	Curriculum
Fall 2011	Technology
Spring 2012	Evaluation
Fall 2012	Equity
Spring 2013	Leadership
Fall 2013	Modeling
Spring 2014	Teaching Aids
Fall 2014	Special Students

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Bourbaki at Seventy-Five: Its Influence in France and Beyond

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Teachers College Columbia University

Bourbaki, a professional mathematical society in France, was established formally in 1935. Throughout three quarters of a century, Bourbaki changed mathematics and mathematics education significantly. It was instrumental in the Moderne Mathématique movement that modernized French curricula. Bourbaki also provided philosophical support for the New Math movement in the United States.

The Origin of Bourbaki

Few scholars of intellectual history can imagine that the name of Nicolas Bourbaki, a French general in the Franco-Prussian war, would be adopted later as a pseudonym by a circle of prominent French mathematicians. Under the pen name of Bourbaki, a vast body of mathematics textbooks was published that shaped the landscape of mathematics research and more importantly championed a global movement in mathematics education. Bourbaki provided philosophical support behind the New Math movement. Its members were closely connected with the New Math movement that shook mathematics education all over the world.

The origin of the renowned mathematics circle Bourbaki can be traced to a meeting of five French mathematicians in 1934 in *Capoulade*, a café in the Latin Quarter of Paris. The five mathematicians, Jean Dieudonné Henri Cartan, Claude Chevalley, André Weil, and Jean Delsarte, were dissatisfied with the standard analysis textbook *Traité d'Analyse* by E. Goursat used in French universities (Borel, 1998). *Traité* lacked the current developments in Zermelo Fraenkel set theory upon which modern analysis rests, and it also omitted vector analysis, which is essential in mathematical physics. Most critically, *Traité* was not written with mathematical rigor—a gold standard the five attendees held in the highest regard. The group rejected the free flowing intuitions once prized by Poincaré. Rigor was not only a criterion for scientific exposition; members of Bourbaki were convinced that mathematics itself should rest on a bedrock of unshakable fundamental principles impervious even to the most unforgiving critics. A new college-level textbook on analysis was needed (Senechal, 1998). The attendees also agreed that the mathematical gap between a student in lycée, the French high school, and that of a university mathematics student was widening because of the rapid development in analysis in the late 19th and early 20th century. From the outset, the founders agreed, their writing must be modern above all. Only a modern analysis textbook can close the gap for interested lycée students who want to pursue a career in mathematics. As the meeting lingered, an ambitious criterion for the upcoming book was adopted: the textbook must be able to accommodate all readers: professional academicians, university students, and serious amateurs with a persistent curiosity (Beaulieu, 1993).

Given such a disparate set of challenges, the Bourbaki founders discovered a unique solution: a self-contained and rigorous analysis textbook from scratch, one that is built on elementary set theory. Though elegant, this solution created a structure emblematic of Bourbaki's published works: a linearized ordering of contents in which a new mathematical concept can only reference concepts that appear in earlier chapters. In its striving for logical rigor, intuition is stymied summarily. This choice of rigor over intuition became the signal criticism against Bourbaki's publications in all coming decades, one that always haunted Bourbaki—and one that Bourbaki never recanted. In an interview in 1997, Pierre Cartier, a later member of Bourbaki, acknowledged that the series of publications is organized more as an encyclopedia rather than textbooks, notwithstanding the muted irony that it was the inadequacy of the *textbook Traité d'Analyse* that precipitated the very first meeting in 1934 (Senechal, 1998).

Over the next few meetings, another complexity emerged: a single textbook on analysis was not sufficient. At the time, analysis and mathematics in general had advanced to a point where previously distinct branches were beginning to overlap. For example, analyticity is usually a topic discussed in complex analysis, but the set of analytic functions on the complex unit circle $\{f \mid f: S^1 \to S^1\}$ forms a group under composition. Bourbaki members determined that six topics were essential: set theory, algebra, topology, functions of one real variable, topological vector spaces, and integration. Each topic warranted a separate book. Each book was assigned to a single member who was to write the first and revised drafts. André Weil was elected the editor-in-chief of all six books in addition to his own topic book. All volumes, current and future ones, hold the general title of *Éléments de Mathématique*. A subtitle identifies the specific contents of each book: algebra, real analysis, or topology (Borel, 1998). The six chosen topics, though not exhaustive, encompass the majority of pure mathematics. In the dim light of the Parisian café, a vast and overreaching mathematics empire was beginning to emerge. The first volume of the *Éléments de Mathématique* was published in 1935. In the same year, the small circle of mathematicians elected to name their group *Association des collaborateurs de Nicolas Bourbaki*—Bourbaki was formally born.

Bourbaki and Mathematics Education

Philosophically, Bourbaki espoused a view akin to Hilbertism—one based on a system of axioms, from which a second stratum of theorems can be derived. Structure occupies the centerpiece in Bourbaki's philosophy of mathematics. Structure assumes two dimensions. First, structure means mathematical knowledge, and indeed that of all natural sciences, possesses subtle and deep structures. As such, knowledge should never be conceived as an orderless collection of facts. Second, because knowledge possesses such structure, the correct pedagogical approach is to elucidate the fundamental structures behind the endless examples. The underlying ideas must be elevated above the examples that illustrate them. That is, the first structure *informs* the second structure. Bourbaki strives for the "why" in place of the "how" (Corry, 2007). In this philosophical and pedagogical perspective, Bourbakists are neo-Platonists. Bourbaki's conviction in the universality of an axiomatic system, grounded in set theories, characterizes both its writings and their philosophy of science. The belief in structure would influence the movers of the New Math movement decades later and transform the landscape of mathematics education significantly.

As Bourbaki grew, its members also changed. Mandatory retirement is required upon reaching age fifty. Young mathematicians were observed carefully in a "trial period" in which their suitability was evaluated. The true criteria for membership were never revealed even to other eminent mathematicians outside the circle. Most mathematics historians posit that a shared vision of structure and rigor in style are crucial requirements for prospective members. The first generation Bourbakists, or the founders, screened their successors meticulously, as did the second generation Bourbakists to the third. Breadth and depth of mathematical knowledge are only the beginning requirements. As such, membership within a generation registers only in single digits. The list of Bourbaki members is a "who's who" in 20th century mathematics. Second, third, and fourth generation Bourbakists included four Fields medalists: Laurent Schwartz, Jean Pierre Serre, Alexander Grothendieck, and Pierre Deligne. Bourbaki also admitted non-French members: Polish topologist Samuel Eilenburg of Columbia University, American number theorist Serge Lang of Yale University, and Swiss algebraist Armand Borel of the Institute of Advanced Study are few examples. As the new members were admitted to the circle, new volumes to the *Éléments* were added. The younger members more than updated the encyclopedic Bourbaki collections; they also disseminated Bourbaki's visions on mathematics education eagerly to the scientific community around the world (Senechal, 1998).

The Bourbaki Influence in Education in France

As an intellectual society native to France, most members of Bourbaki held prominent academic positions in elite French universities. Yet the political realities of Europe handicapped Bourbaki's mission to reform mathematics education in France for several decades. Mathematics historians must be cautioned that the outbreak of WWII in 1940, five years after the birth of Bourbaki, halted most research activities. After WWII, economic recovery was the top national priority for France. Curriculum reform was only given a cursory glance. Additionally, France does not enjoy local independence as does the United States; any educational reform entails a massive governmental effort from top down, painstakingly implemented in every classroom across the country.

In 1967, the French government organized a panel of 18 mathematicians and educators headed by the differential geometer André Lichnerowicz to make recommendations on mathematics curriculum. Though not a Bourbaki member, Lichnerowicz studied under Eli Cartan, the father of Henri Cartan, a first generation Bourbakist. Lichnerowicz was a staunch devotee of the ideals of Bourbaki. The Lichnerowicz Commission made sweeping changes in the French mathematics curriculum both in contents and in pedagogy. Deeply committed to the principle of structure behind the problems, the Lichnerowicz Commission recommended a revised curriculum based on set theory with an early introduction to mathematical structures. Computation, once the main staple of pre-college mathematics education, was de-emphasized. New textbooks were commissioned to reflect the shift. It is important to note that this reform is referred to as the Moderne Mathématique movement as opposed to the New Math movement in the United States.

As a result, France experienced landslide changes in secondary mathematics education. Set theories, complex numbers, probability, and other abstract concepts are introduced earlier to students. One must be cautioned that it is not the early introduction of these advanced concepts that changed the French secondary mathematics curriculum; it is the concept of structure that stood at the core of the new curriculum. Complex analysis, for example, has existed since the time of Gauss. Simply introducing complex analysis into the secondary education curriculum is not the objective of the Lichnerowicz Commission. The new French textbooks commissioned by the panel reflected the breathtaking depth of Bourbaki's vision of structure. In many respects, the curriculum reform in France embodies the spirit of Bourbaki more closely than the reform in the United States. Take the example of the definition of complex numbers. The new French textbooks contain a Bourbaki definition cumbersome at first glance: a complex number (a,bi) is defined as a matrix

$$M(a,b) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Standard American textbooks define a complex number (a,bi) as an ordered subset of R^2. The French Bourbaki definition stands out as an ungainly oddity compared to the streamlined American definition. The clean-cut American definition even alludes to the ring structure of R^2 with coordinatewise addition and multiplication as its operations:

$$(a,bi) + (c,di) = (a+c,(b+d)i) \in \mathbb{R} \oplus \mathbb{R}i \cong \mathbb{R}^2.$$

The French Bourbaki definition also passes the additive test, but without the American alacrity:

$$\begin{aligned}(a,bi) + (c,di) =& M(a,b) + M(c,d) \\ =& \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \\ =& \begin{pmatrix} a+c & -b-d \\ b+d & a+c \end{pmatrix} \\ =& M(a+c,b+d) \\ =& (a+c,(b+d)i).\end{aligned}$$

But with multiplication, the Bourbaki definition, constructed with rich structures in mind, begins to show its superiority:

$$\begin{aligned}(a,bi) \times (c,di) =& M(a,b) \times M(c,d) \\ =& \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \times \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \\ =& \begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix} \\ =& M(ac-bd,ad+bc) \\ =& (ac-bd,(ad+bc)i).\end{aligned}$$

whereas the American definition, though prized for its simplicity at first, begins to show its weaknesses as it relies on the alluded multiplicative structure of the ring R^2:

$$(a,b) \times (c,d) = (ac,bd) \neq (ac-bd,(ad+bc)).$$

The Bourbaki definition holds another edge over the American definition. With complex numbers, modulus $|(a,bi)| = \sqrt{(a^2 + b^2)}$ is a central concept. The square of the modulus removes the square root: $a^2 + b^2 = |(a,bi)|^2$. The Bourbaki definition lends itself seamlessly to the square of the modulus function:

$$\begin{aligned}&|\ (a,bi)\ |^2 \\ &= \det M(a,b) \\ &= a^2 - (-b^2) \\ &= a^2 + b^2\end{aligned}$$

On the other hand, the American definition $(a,bi) \in \mathbb{R} \oplus \mathbb{R}i \cong \mathbb{R}^2$ does not in any way elicit modulus. The use of the determinant function in defining the moduli of complex numbers shows an important aspect of modulus as a function from the complex plane to the real numbers: $|\ |$: C → R; x+yi → $x^2 + y^2$. Notice the determinant function, taken as the function that computes the square of the modulus, is a polynomial in two variables. As such, the square of the modulus is a continuous function. This is a critical idea used repeatedly later both in and outside of complex analysis in concepts such as the Banach fixed point theorem or the existence and the uniqueness of solutions to ordinary differential equations. The strength of Bourbaki is felt keenly and powerfully.

The Bourbaki Influence in Education in the U.S.

With Bourbaki's increasing membership of American mathematicians in the second and third generations, the French intellectual circle began to influence American curriculum design. In America, Bourbaki provided philosophical support behind the New Math movement. The key movers of the New Math movement in America are university mathematicians who wanted school mathematics curricula to reflect the same modern cast that university mathematics courses were acquiring. The purpose was to reduce the discontinuity between pre-college and collegiate mathematics as well as to encourage more students to pursue the study of mathematics (Kilpatrick, 2010).

It is certain that the educators behind the New Math movement believed deeply in the underlying structure in mathematics. Like their French counterparts, New Math recommended a curriculum where computations are de-emphasized.

The earliest New Math reform came from the University of Illinois's Committee on School Mathematics (UICSM) chaired by Max Beberman in 1951. A graduate of Teachers College Columbia University, Beberman professed a staunch belief in "discovery learning." UICSM designed textbooks that incorporate guided discovery into many lessons. The chapters were written such that the students recognize the patterns emerging from the mathematical expressions, but the generalized mathematical principles were not given explicitly until later chapters. Studies have shown that the Beberman textbooks work well with educators who are trained in discovery learning.

Another proponent of New Math was the topologist Edward Begle, director of the School Mathematics Study Group (SMSG). A seasoned educator, Begle conducted fundamental studies in mathematics epistemology. Under Begle's leadership, SMSG created and implemented New Math curricula for primary and secondary schools in the '60s. Numerous other professional organizations, such as the Minnesota School Science and Mathematics Center or the Ball State Project, also supported New Math. NCTM organized its own Secondary School Curriculum Committee to discuss implementation of the New Math Curriculum. By the middle of the 1960s, many high schools in the United States adopted New Math curricula.

New Math attracted as many supporters as detractors. Under New Math, students' abilities to execute basic arithmetic computations suffered. Many educators and parents were critical of the abstract contents introduced into elementary and secondary mathematics textbooks that bear little resemblance to the pragmatic knowledge readily useful outside the classroom. By the end of '70s, the New Math movement ended. In the ensuing decades, the United States shifted between the Bourbaki-inspired New Math curriculum and the traditional curriculum.

Bourbaki Today

Three more books were added to the original six books discussed in the first Bourbaki Congress. Today, *Éléments* totals forty volumes. The last of the Bourbaki series, book IX on spectral theory, was published in 1983. The annual Bourbaki Congress trickled to a stop briefly after the last publication. Since then, Bourbaki has remained out of the scientific spotlight.

2010 marks the seventy-fifth birthday of the Bourbaki group. Intellectual historians laud its immense contributions to mathematics, to mathematics education, and to the scientific community at large. Looking back from 2010 to 1935, from one textbook to forty volumes, from a group of five disaffected mathematicians to a global movement in mathematics education, Bourbaki has journeyed a long way. Newcomers to mathematics gaze at the numerous volumes of *Eléments* with bewilderment, shocked by the prolific energy of "the extraordinary Dr. Bourbaki," while senior mathematicians smile quietly in the back row, reminiscing about the heyday of a circle of mathematics illuminati that moved and is still moving mathematics and mathematics education at age 75.

References

Beaulieu, L (1993). A Parisian café and ten proto-Bourbaki meetings. *The Mathematical Intelligencer, 15*(4), 27–35.

Begle, E., et al. (Eds.) (1966). *The role of axiomatics and problem solving in mathematics*, (Report of a sub-conference at the quadrennial International Congress of Mathematicians in Moscow). Conference Board of the Mathematical Sciences. Washington, DC: Ginn.

Borel, A. (1998). Twenty-five years with Nicolas Bourbaki. *Notices of the AMS, 45*(3), 373–380.

Cha, I. (1999) Mathematical and pedagogical discussions of the function concept. *Journal of the Korea Society of Mathematical Education, 3*(1), 35–56.

Corry, L. (1998). The origins of eternal truth in modern mathematics: Hilbert to Bourbaki and beyond. *Science in Context, 12*, 137–183.

Corry, L (2007), Axiomatics between Hilbert and the New Math: Diverging views on mathematical research and their consequences on education. *International Journal for the History of Mathematics Education, 2*(2), 21–37.

Kilpatrick, J. (2010), personal communication.

Mashaal, M. (2006). Bourbaki: A secret society of mathematicians. Providence, RI: American Mathematical Society.

Miller, J. (1990), What ever happened to New Math? *American Heritage, 41*(8).

Senechal, M. (1998). The continuing silence of Bourbaki—An interview with Pierre Cartier. *Mathematical Intelligencer, 20*(1), 22–28.