Journal of Mathematics Education at Teachers College

Fall – Winter 2011

A CENTURY OF LEADERSHIP IN MATHEMATICS AND ITS TEACHING

The *Journal of Mathematics Education at Teachers College* is a publication of the Program in Mathematics and Education at Teachers College Columbia University in the City of New York.

Dr. Robert Taylor was selected by the Teachers College sponsored Teachers for East Africa program to teach mathematics of Uganda's Makerere University. He returned to TC as an instructor in the Department of Mathematics, Statistics, and Computing in Education where he developed an innovative programming language (FPL) intended to introduce educators to the then - new field of computer programming. His seminal work entitled *Computers: Tutor, Tool, Tutee* led to leadership in the new field of computers in education. Dr. Taylor completed 33 years as a member of the Teachers College faculty in 2009.

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Aims and Scope

The *JMETC* is a re-creation of an earlier publication by the Teachers College Columbia University Program in Mathematics. As a peer-reviewed, semi-annual journal, it is intended to provide dissemination opportunities for writers of practice-based or research contributions to the general field of mathematics education. Each issue of the *JMETC* will focus upon an educational theme. The themes planned for the 2012 Spring-Summer and 2012 Fall-Winter issues are: *Evaluation* and *Equity*, respectively.

JMETC readers are educators from pre K-12 through college and university levels, and from many different disciplines and job positions—teachers, principals, superintendents, professors of education, and other leaders in education. Articles to appear in the *JMETC* include research reports, commentaries on practice, historical analyses and responses to issues and recommendations of professional interest.

Manuscript Submission

JMETC seeks conversational manuscripts (2,500-3,000 words in length) that are insightful and helpful to mathematics educators. Articles should contain fresh information, possibly research-based, that gives practical guidance readers can use to improve practice. Examples from classroom experience are encouraged. Articles must not have been accepted for publication elsewhere. To keep the submission and review process as efficient as possible, all manuscripts may be submitted electronically at www.tc.edu/jmetc.

Abstract and keywords. All manuscripts must include an abstract with keywords. Abstracts describing the essence of the manuscript should not exceed 150 words. Authors should select key words from the menu on the manuscript submission system so that readers can search for the article after it is published. All inquiries and materials should be submitted to Ms. Krystle Hecker at P.O. Box 210, Teachers College Columbia University, 525 W. 120th St., New York, NY 10027 or at JMETC@tc.columbia.edu

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The Frame Game

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In this article, the authors explore framing, a non-multiplicative technique commonly employed by students as they construct similar shapes. When students frame, they add (or subtract) a "border" of fixed width about a geometric object. Although the approach does not yield similar shapes in general, the mathematical underpinnings of framing are noteworthy. Using Interactive Geometry Software (IGS), teachers can explore mathematics behind the framing technique in greater depth while addressing student misconceptions and over-generalizations associated with the approach in classroom settings.

Keywords: student misconceptions, geometry, proportionality, technology, similarity.

Introduction

During our first years as introductory geometry teachers, we were genuinely surprised by misconceptions that our students exhibited as they attempted to identify and construct similar shapes. "Similar shapes? That should be easy!" we thought. "Similar objects have the same shape. That's it. What's so hard about recognizing objects with the same shape?" However, through our teaching experiences with students in entry-level courses, we've come to appreciate the complexity embedded within similarity tasks. Growth tasks—geometric situations involving the scaling of objects—have proved particularly challenging for younger students (Lamon, 1993) and secondary students alike. Success with growth tasks requires students to apply multiplicative strategies and proportional reasoning; however, struggling students tend to approach such problems using non-multiplicative methods (Lamon, 1993). Consider, for instance, the Similar Rectangles Task, a growth task shown in Figure 1. Successful students use multiplicative reasoning to determine that the rectangles in Figure 1 are not similar. For instance, since the ratios of heights and widths from large to small rectangle are not equivalent ($5/3 \neq 15/13$), corresponding sides of the rectangle are not proportional. In other words, there is no size change transformation that maps one rectangle onto the other.

In spite of this, a significant portion of our students copy the smaller rectangle within the larger one, employing a non-multiplicative technique that we refer to as framing. An example of the approach is illustrated in Figure 2. Intrigued by the popularity of framing as a method for identifying and constructing similar shapes, we decided to explore its mathematical underpinnings in more depth, hopeful that such study would provide insight regarding the attractiveness of framing for our students as well as possible teaching strategies to help our students overcome the urge to use the technique inappropriately. Findings from our investigation have informed our teaching—and thus our students' understanding of similarity and proportionality. In the remainder of this paper, we examine the framing technique more formally

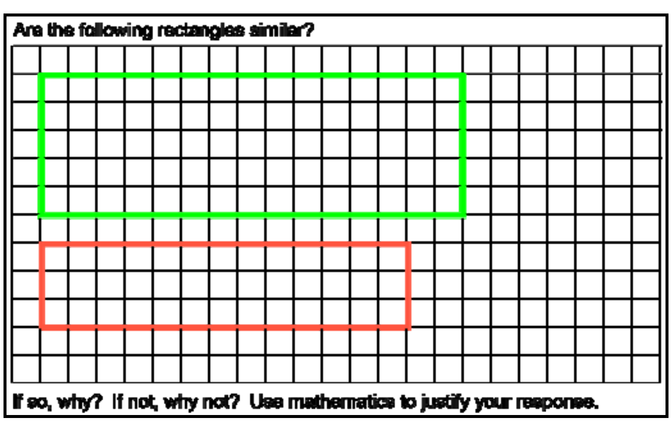

Figure 1. Similar Rectangles task

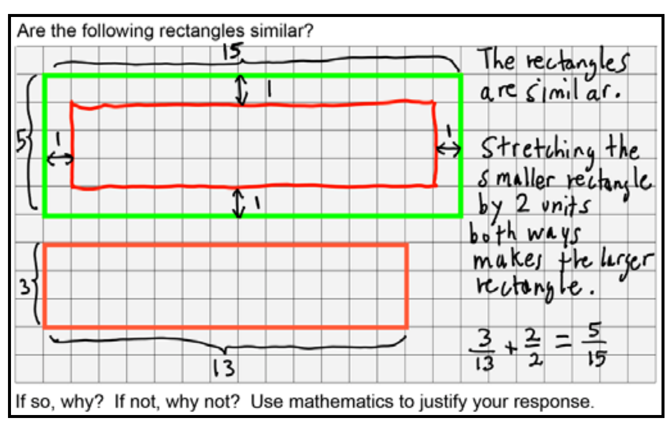

Figure 2. Incorrect response using a framing argument

and share a model for incorporating Interactive Geometry Software (IGS) to address student misconceptions associated with framing.

The Framing Technique

We define framing as the process of adding (or subtracting) a “border” of fixed width about a geometric object. Why is this technique so convincing for students and so difficult for teachers to extinguish? To answer such a question, we need to determine shapes that yield similar images when framing is applied (if any exist). To help extinguish use of the technique, we also need to find various shapes that yield obviously dissimilar objects when framing is applied. In this way, we hope to craft opportunities for students to examine and evaluate the strategy autonomously.

To our initial surprise, framing works well for a variety of familiar shapes. Consider, for instance, a square with side s. Attaching paper strips of width w along each edge of the square yields a larger square, as shown in Figure 3. The paper strips yield sides parallel to those in the original shape, thus right angles are preserved. Furthermore, sides of the resulting frame (shown as dotted segments in Figure 3) are proportional to the sides of the original square, at a ratio of s+2w to s. In general, squares are similar to squares. Hence, the resulting framing square is similar to the original framed square. A general argument reveals that the framing process yields similar figures for any regular polygon. For instance, the framed and framing pentagons in Figure 4 are similar—so, too, are the regular hexagon and regular octagon pairs. Because student work with geometric shapes, particularly in the early grades, focuses on regular figures, it is not surprising that framing is an attractive (albeit incorrect) technique for entry-level students. In the following section, we provide a general argument confirming framing as a valid technique for producing similar regular polygons.

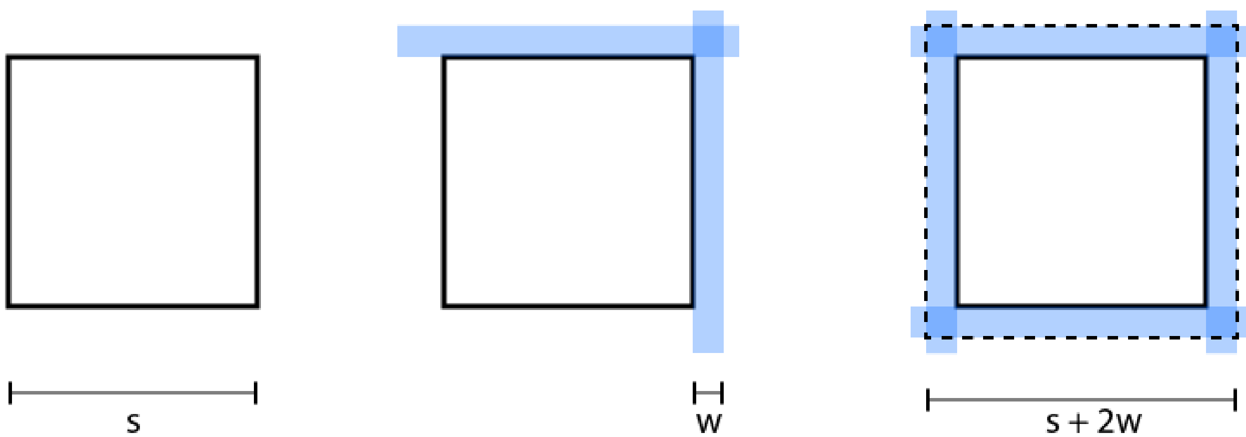

Figure 3. Framing applied to a square with side length s units

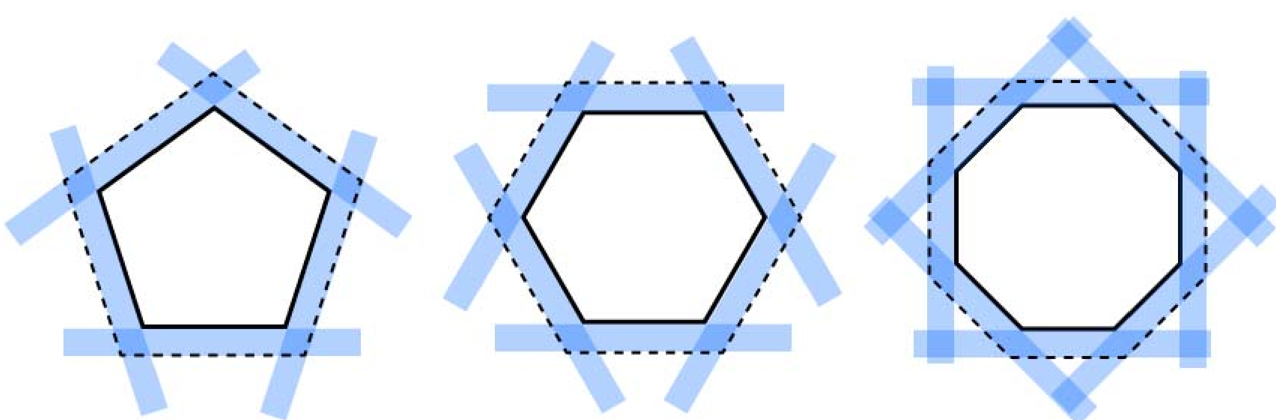

Figure 4. The framing process applied to various regular polygons

General Proof of Applicability of Framing Method for Regular Polygons

Corresponding angles are congruent. Given a regular n-gon with side length s, we first show that the angles of the resulting shape obtained from framing are congruent to corresponding angles in the original n-gon. Begin by considering consecutive, congruent framing strips with fixed width w intersecting at arbitrary vertex W. The overlapping strips form quadrilateral WAYB. As illustrated in Figure 5, we let α represent the measure of the interior angle of the regular polygon at W; and we let γ represent the measure of angle AYB. We show that α = γ. As Figure 6 illustrates, consecutive overlapping strips can be cut into rectangles that intersect at precisely one point, namely W. Points X and Z are vertices of the cut rectangular strips.

We let β represent the angle formed by cut strips intersecting at W. Note that $\beta = 360^\circ - (\alpha+90^\circ+90^\circ) = 180^\circ - \alpha$. Since WXYZ is a quadrilateral, $\beta+90^\circ+90^\circ+\gamma=360^\circ$. Substituting $180^\circ - \alpha$ for β in the previous equation, we note that $(180^\circ - \alpha)+90^\circ+90^\circ+\gamma=360^\circ$ which yields α = γ. Employing an analogous argument at each of the other n-1 vertices, we conclude that interior angles of the original framed polygon are congruent to corresponding angles of the framing polygon.

Corresponding sides are proportional. Next, we argue that sides of the framing polygon are proportional to sides of the original regular n-gon. As Figure 7 suggests, the framing strips may be cut to form congruent rectangles. The length of each cut framing strip equals s, the side length of the original regular n-gon.

At arbitrary vertex W, quadrilateral WXYZ is split into two right triangles by diagonal WY, as shown in Figure 8. By reflexivity, segment WY is congruent to itself. Furthermore, segments WX and WZ are congruent since the width of each framing strip is the same. By the Pythagorean Theorem, segments XY and ZY have the same length—call this x units. Using an analogous argument at each vertex of the original n-gon, we conclude that the length of each side of the framing n-gon is s+2x units. Hence, the corresponding sides of the framing polygon are proportional to sides of the original regular n-gon with ratio s+2x to s.

Since (1) the angles of the shape resulting from framing are congruent to corresponding angles in the original n-gon and (2) the corresponding sides of the framing n-gon are proportional to sides of the original regular n-gon, we conclude that the framing approach yields similar shapes for regular n-gons.

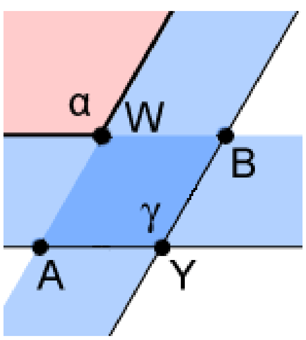

Figure 5. Consecutive, overlapping framing strips for arbitrary, regular n-gon

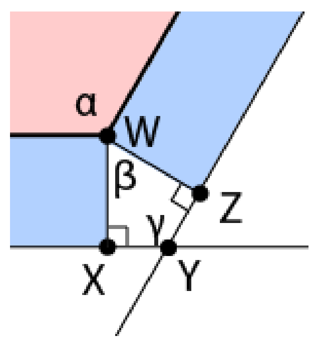

Figure 6. Quadrilateral formed consecutive cut, rectangular framing strips

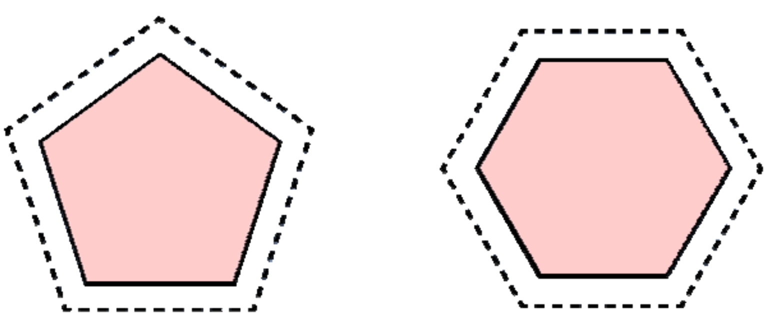

Figure 7. Cut framing strips for several regular n-gons

Addressing Student Misconceptions Associated with Framing

Unfortunately, the framing process does not yield similar shapes for non-regular figures and numerical arguments are often too abstract to be convincing. We have found that it is more productive to appeal to a visual argument since, in our experience, students have some intuition about what proportional shapes should look like and can sense when distortion is present. For example, when using the framing technique to scale a heart shape, Emerald generated a series of concentric hearts. She wrote at the bottom of her paper, “I know its not right.” It occurred to us that Emerald’s use of visual evidence to judge the similarity of the constructed hearts was a powerful tool; a tool that we wanted to develop more formally in the classroom. Iterative framing, illustrated in Figure 9, challenges students’ additive strategies; however, additive thinking is remarkably resilient (Hart, 1988). In paper-and-pencil scaling tasks such as the Two Hearts task, students are often more willing to challenge their own drawing skill than the mathematics behind the strategy. Even after students explore framing and identify shapes on which it would and would not work, it may still be unclear that framing is an additive strategy. Technology, particularly IGS can help us overcome this hurdle by allowing students more freedom to test mathematical conjectures in a precise environment.

Based on these experiences, we developed a three-stage approach to using interactive geometry software (IGS) to investigate the framing technique and support students in moving away from additive thinking. First, we formalize and refine students’ visual intuitions by using IGS to **test** the similarity of two pre-existing geometric objects. Second, we incorporate iteration into our lesson to help students **confront** deeply-held misconceptions surrounding the validity of framing as a technique for constructing similar shapes. Last, using IGS, students **construct** proportional shapes from provided pre-images and analyze the existing multiplicative relationships. We discuss each step of this process with more detail and illustration in the sections that follow.

Stage 1: Testing the Similarity of Framed Images

In this stage, students are presented with two pre-existing geometric objects and are asked to determine whether the objects are similar. For instance, the sketch depicted in Figure 10 asks students to determine if the grey octagonal borders in the logo are similar. An interactive version of the dynamic sketch is available on-line at http://tinyurl.com/logosketch. Clicking on the “Show slider” checkbox reveals a slider that controls the scale factor of a dilation of the “inner” grey octagon. When the value of the slider is increased, the “inner” octagon grows larger; when the value is decreased, the “inner” octagon shrinks. Through experimentation, students find that a scale factor of 1.6 (depicted in Figure 10, right) produces a dilation of the “inner” octagon that has the same height as the “outer” grey octagon. Since the two shapes have the same height but different widths, students conclude that the dilation and the “outer” octagon aren't similar. Since the dilation is similar to the inner octagon, they conclude that the “inner” and “outer” octagons are not similar.

In the testing stage, IGS offers students several advantages over traditional paper and pencil work. First, the sketches encourage students to consider similarity from a geometric vantage point (as opposed to a purely numerical view often emphasized in worksheet activities). Second, as students change scale factors of dilations within dynamic sketches, they are encouraged to consider similar shapes as pre-image and image of a size-change transformation.

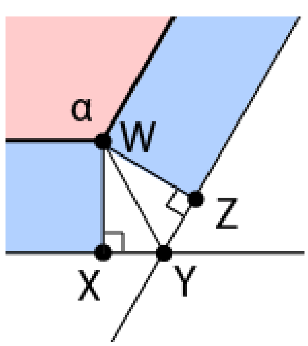

Figure 8. Quadrilateral WXYZ split into two right triangles by diagonal WY

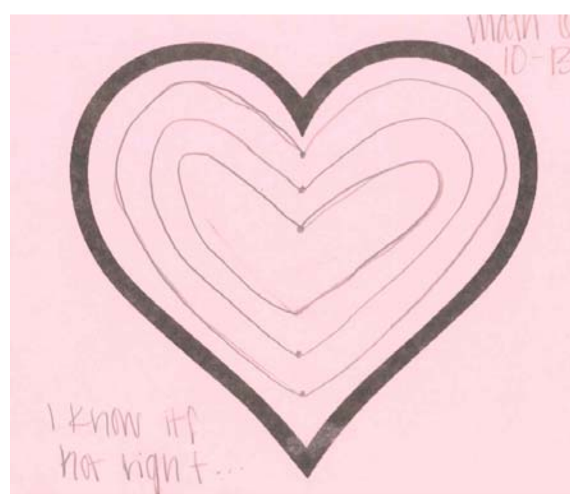

Figure 9. Emerald’s Framed Hearts

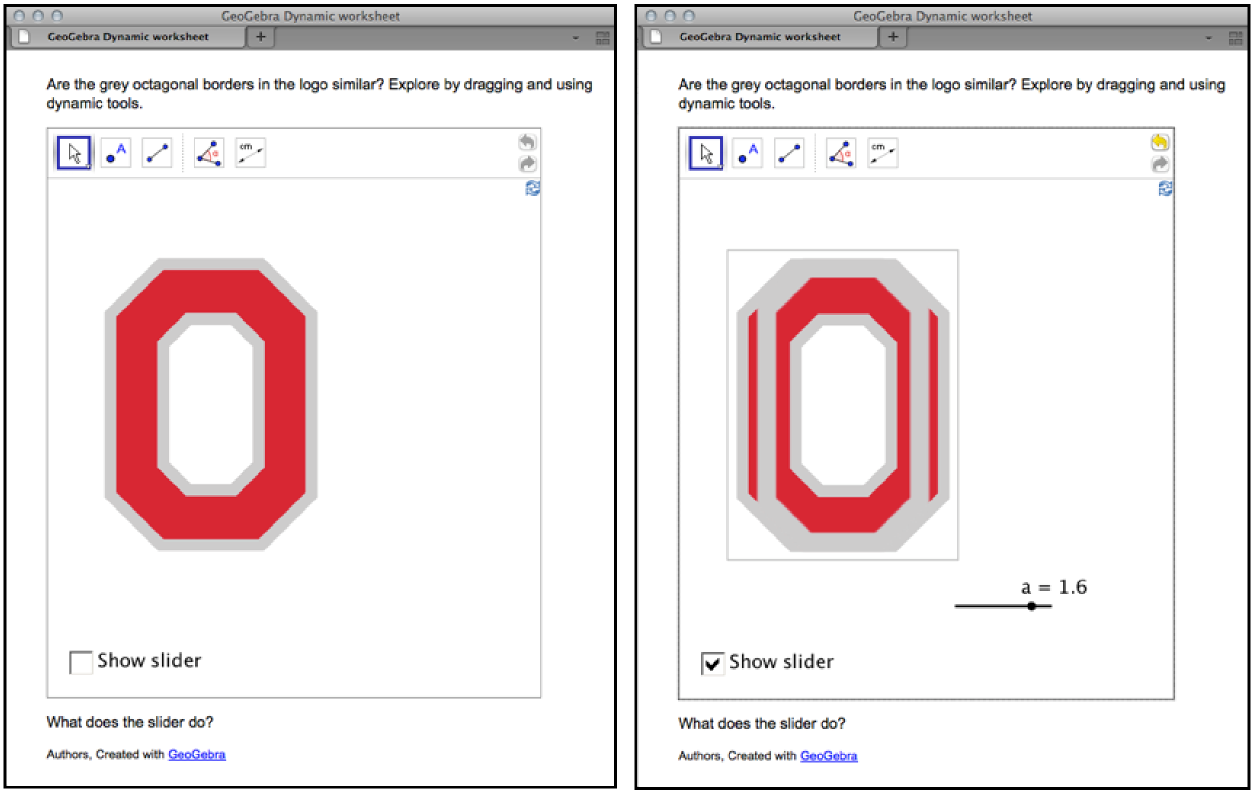

Figure 10. (Left) Dynamic logo investigation; (Right) Logo with dilated "inner" octagonal border

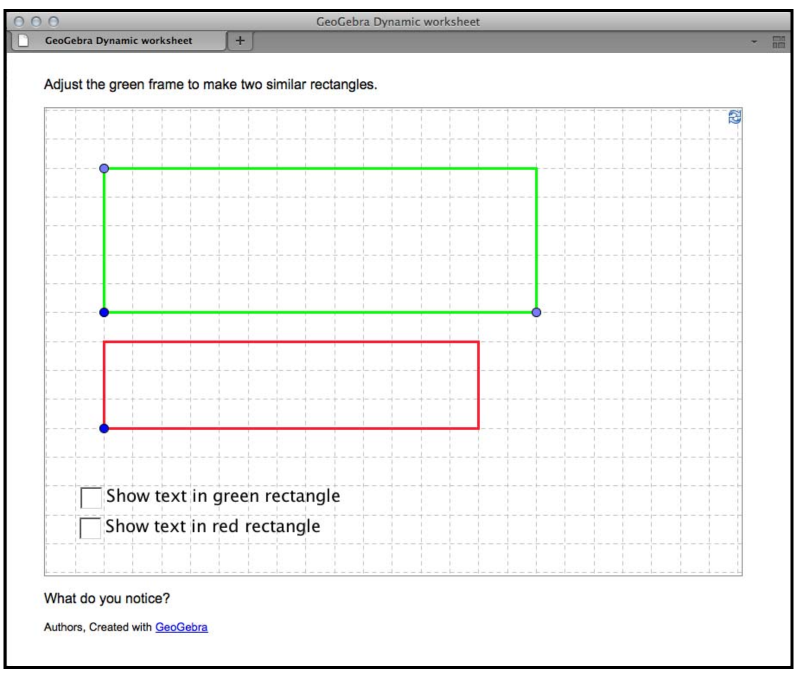

Figure 11. Dynamic version of the Similar Rectangles task

Stage 2: Confronting Misconceptions

When framing is used on non-polygons, it introduces distortion. In Stage 1, we helped students become more aware of that distortion. It is more difficult to detect that distortion when working with simple polygons such as rectangles and triangles. In Stage 2, we challenge students to see more nuanced versions of that distortion and conduct more rigorous mathematical analysis of the technique in two ways. First, placing a graphic object (e.g., text or a digital photo) within a non-square rectangle prior to framing may help students recognize that framing doesn't work in harder-to-recognize shapes. The dynamic sketch in Figure 11 initially shows a 3 x 13 rectangle and 5 x 15 rectangle. An interactive version of the sketch is available online at http://tinyurl.com/affront-sketch.

Using the transformational features of IGS, students drag the lower left vertices to re-position the rectangles as frames. Dragging the remaining vertices, students stretch the framing rectangle (and enclosed graphic) vertically and horizontally in an effort to construct similar shapes. As suggested in Figure 12, clicking on the "show text" checkboxes reveals a graphic element embedded within each rectangle. After some experimentation, many students note that the text in the two rectangles (particularly the o's) appears "different" for any pair of rectangles. Text within the framing rectangle appears to be stretched. Distortion, such as that depicted in Figure 11, provides compelling evidence that framing fails to yield similar rectangles (Cox, 2010).

Second, IGS allows students to apply the framing process repeatedly to an initial rectangle, forming "concentric" shapes (Cox and Edwards, 2011). Such a process is depicted in Figure 13, left. In the sketch, successive frames are two units taller and wider than the next largest shape. Note that with each successive iteration, the outer-most frame becomes more "square-like." Clearly, for this reason, the initial rectangle and

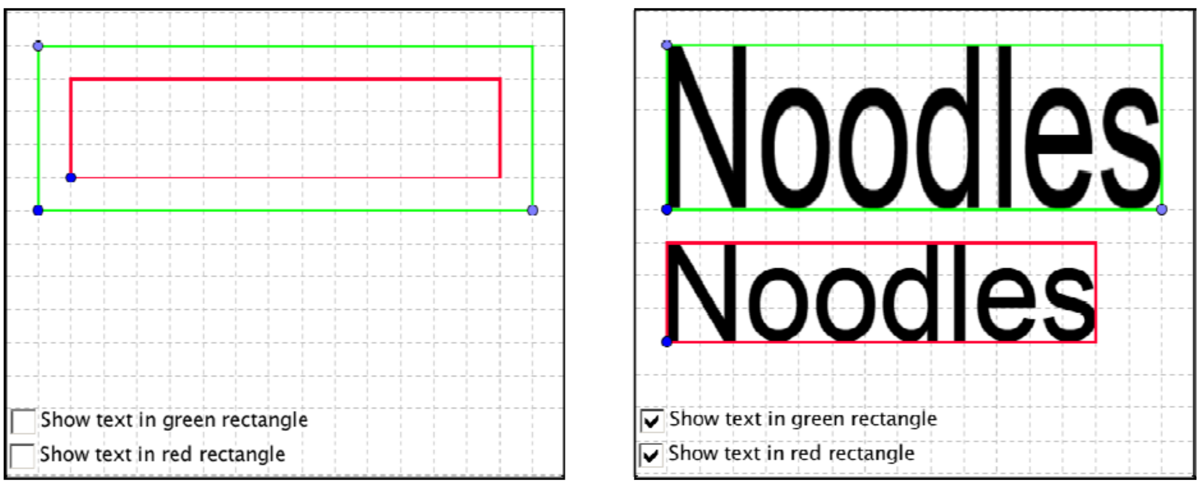

Figure 12. (Left) Dynamic sketch allows rectangles to be repositioned; (Right) Text embedded within rectangles

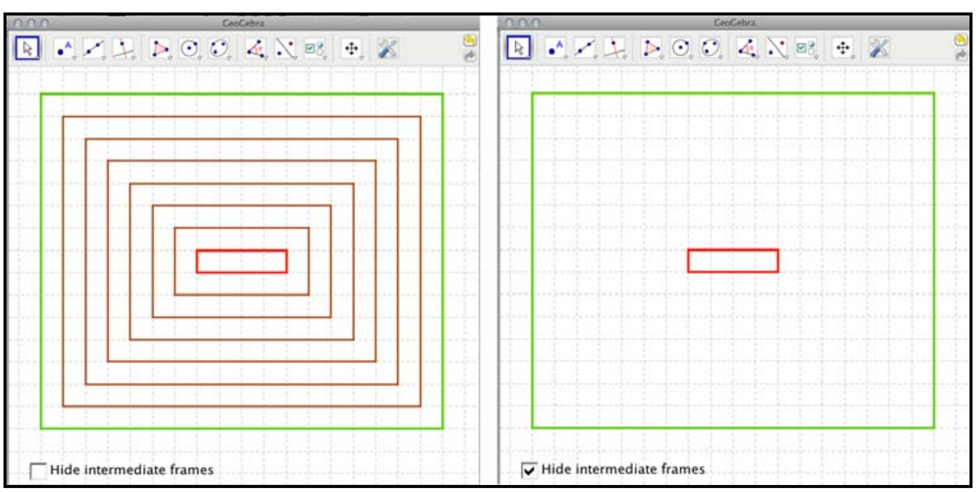

Figure 13. (Left) Iterated framing process; (Right) Sketch with intermediate frames hidden

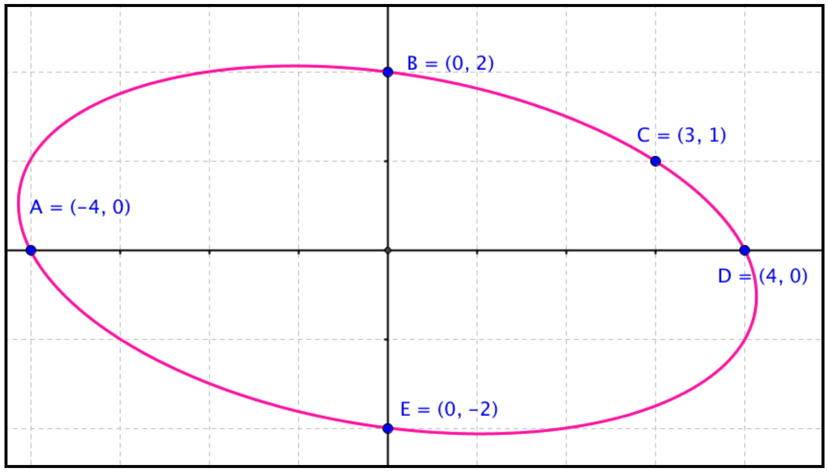

Figure 14. Ellipse passing through points A(-4,0), B(0,2), C(3,1), D(4,0), and E(0,-2).

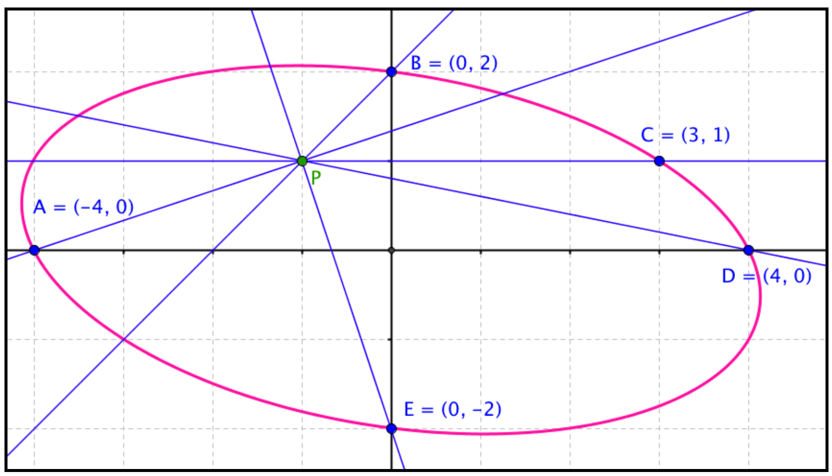

Figure 15. A line is drawn through the center of dilation, P, and each of five points on the pre-image.

outer-most frame are not similar. As Figure 13, right, suggests, hiding intermediate frames within the sketch makes the result more obvious. Since the rectangles in Figure 13 were produced using a framing approach, the process casts doubt on framing as a legitimate strategy for producing similar shapes. An interactive version of the sketch is available online at http://tinyurl.com/repeated-framing.

As the preceding examples illustrate, common misconceptions regarding framing may be addressed using IGS. The software allows sketches that are more precisely and more quickly drawn than possible with paper-and-pencil alone. Moreover, dynamic features of the sketches afford students significant opportunities for experimentation.

Stage 3: Constructing Similar Shapes

Constructivism tells us that students need opportunities to construct their own similar shapes (if we want them to truly grasp similarity). In our classes, students use three basic strategies for constructing similar shapes with IGS:

1. Manual dilation (e.g., choosing center of dilation, drawing lines from key points of pre-images);
2. Scaling and plotting of coordinates stored in lists; and
3. Using a geometric dilation tool.

In this stage, we typically provide students with novel shapes to explore with IGS. From our previous work, we know that the framing technique is a convincing method for constructing similar shapes because the approach

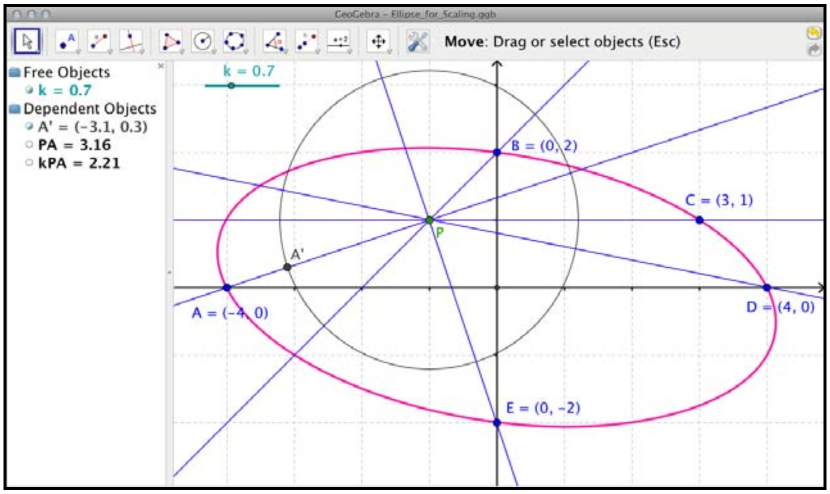

Figure 16. Constructing A', the dilation image of A about center P with scale factor, k.

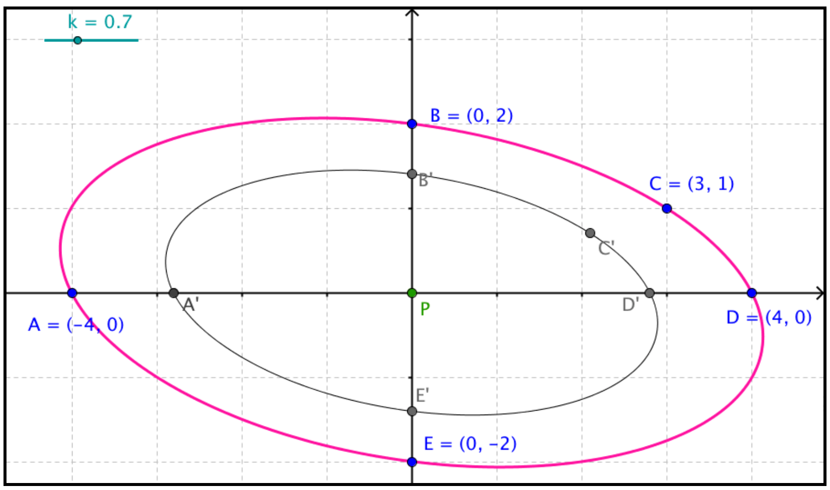

Figure 17. Similar ellipse centered at P passing through points A', B', C', D' and E'.

works for a variety of familiar shapes—namely, regular polygons and circles. If we desire to help students build robust understandings of proportionality—one which rejects the framing technique as a general method for constructing similar shapes—then we need to provide students with non-typical examples for which the framing strategy fails and encourage the development of multiplicative strategies with novel shapes. Consider, for instance, the ellipse passing through points A(-4,0), B(0,2), C(3,1), D(4,0), and E(0,-2) shown in Figure 14. Students are asked to construct one or more ellipses similar to the original using tools available within IGS. We highlight solution methods using the three basic techniques mentioned in the preceding paragraph.

Manual Dilation. Manual dilation techniques within IGS mimic pencil-and-paper methods used for dilating figures using ruler and compass. Students begin by constructing a center of dilation (depicted as point P in Figure 15). Next, lines are drawn from the center to key points on the pre-image (in our case, the five known points A, B, C, D, and E on the original ellipse).

Next, we construct a scale factor (in our case, represented by slider, k). Point A', the dilation of A about P, is constructed using a circle centered at Point P. A' is constructed such that k•PA = PA' with A' on line PA. This is suggested in Figure 16. An analogous method is used to construct dilation images B', C', D' and E'.

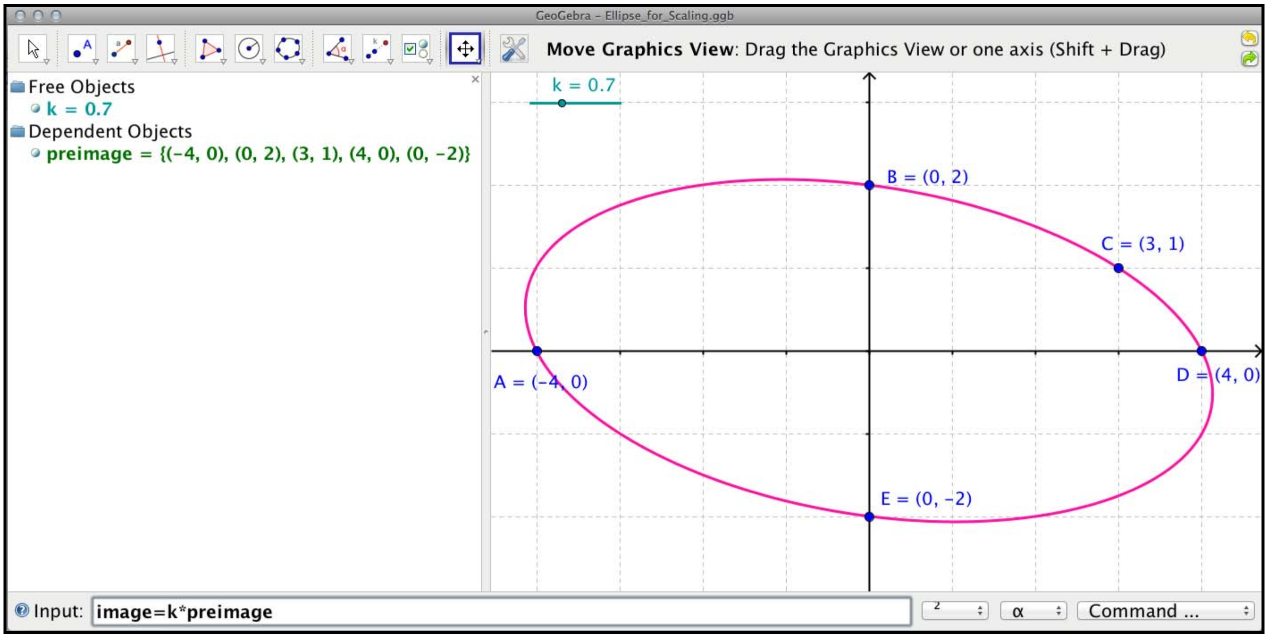

Figure 18. Constructing a dilation using lists in GeoGebra

To complete the manual dilation, as shown in Figure 17, lines are hidden. Using conic construction tools, a scaled ellipse is constructed through points A', B', C', D' and E'. Point P and slider k may be manipulated, allowing students to investigate various similar ellipses with different centers and scale factors.

Scaling and Plotting. Alternatively, dilation images of points A, B, C, D, and E can be constructed using lists. First, a list of pre-image points, pimage, is created with the command pimage = {A,B,C,D,E} in the GeoGebra input bar. Next, the list of image points, image, is created by premultiplying the pre-image points with scale factor, k, as shown in Figure 18. In addition to creating a list of image points dilated about the origin with scale factor k, the command Image = k*Pimage also plots these points. Using conic construction tools, a scaled ellipse is constructed through points A', B', C', D' and E'.

Geometric Dilation Tool. Lastly, students can construct similar ellipses using the geometric dilation tool within GeoGebra. First, center of dilation, P, is constructed along with scale factor (in our case, represented by slider, k). With the dilation tool selected, students click on the pre-image ellipse and center of dilation and then specify k as the scale factor. The results appear identical to those in Figure 17.

We've found IGS-based dilation exercises helpful for encouraging students to consider the construction of similar shapes as a multiplicative process. Manual dilation and scaling lists both require students to multiply objects by scale factor k. Using dynamic capabilities of the software, students may drag points to explore countless similar shapes with the click of a mouse. As students construct similar versions of irregular shapes using each of the three methods, they recognize that in no case is the resulting figure a frame of the original. Furthermore, once students represent scaling in the language and symbolism of dilation, it is possible to return to the framing strategy and examine its additive structure more carefully. After a more conceptual approach, the numeric arguments become more convincing and less abstract.

Conclusion

In this article we have begun to explore similarity from a new perspective that incorporates novel figures (e.g., logos, oblique ellipses, shapes with graphical elements) and IGS technology. This perspective moves us beyond problems that seek to boil down the concepts of scale to rote numerical procedures toward lessons that capitalize on existing experience and understanding that students already have. The tasks presented here have been used to further what students understand, but also to help them rigorously and scientifically challenge it. In our experience, children and adults alike are genuinely surprised by their exploration of the framing strategy and are interested in identifying the conditions under which framing could work. Perhaps the biggest benefit to this work is the opportunity to capitalize on student curiosity and the element of mathematical surprise.

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Call for Papers

The "theme" of the fall issue of the *Journal of Mathematics Education at Teachers College* will be *Evaluation*. This "call for papers" is an invitation to mathematics education professionals, especially Teachers College students, alumni and friends, to submit articles of approximately 2500-3000 words describing research, experiments, projects, innovations, or practices related to evaluation in mathematics education. Articles should be submitted to Ms. Krystle Hecker at JMETC@tc.columbia.edu by January 21, 2012. The spring issue's guest editor, Ms. Heather Gould, will send contributed articles to editorial panels for "blind review." Reviews will be completed by February 1, 2012, and final drafts of selected papers are to be submitted by March 1, 2012. Publication is expected by April 15, 2012.

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This *Call for Volunteers* is an invitation to mathematics educators with experience in reading/writing professional papers to join the editorial/review panels for the spring 2012 and subsequent issues of *JMETC*. Reviewers are expected to complete assigned reviews no later than 3 weeks from receipt of the manuscripts in order to expedite the publication process. Reviewers are responsible for editorial suggestions, fact and citations review, and identification of similar works that may be helpful to contributors whose submissions seem appropriate for publication. Neither authors' nor reviewers' names and affiliations will be shared; however, editors'/reviewers' comments may be sent to contributors of manuscripts to guide further submissions without identifying the editor/reviewer.

If you wish to be considered for review assignments, please request a *Reviewer Information Form*. Return the completed form to Ms. Krystle Hecker at hecker@tc.edu or Teachers College Columbia University, 525 W 120th St., Box 210, New York, NY 10027.

Looking Ahead

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Fall 2012	Equity
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