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TABLE OF CONTENTS

PREFACE

- v William McGuffey, *Teachers College, Columbia University*
Kimberly Barba, *Teachers College, Columbia University*

ARTICLES

- 1 **Some Thoughts on Educating More Able Students**
Geoffrey Howson, University of Southampton, England
- 7 **A Comparison of Mathematics Teachers' and Professors' Views on Secondary Preparation for Tertiary Calculus**
Carol Wade, The College at Brockport, State University of New York; Gerhard Sonnert, Harvard University; Philip Sadler, Harvard University and the Smithsonian Institute; Zahra Hazari, Florida International University; Charity Watson, Clemson University
- 17 **The Legacy Continues: "The Test" and Denying Access to a Challenging Mathematics Education for Historically Marginalized Students**
Richard Kitchen, University of Denver; Sarah Anderson Ridder, University of Denver; Joseph Bolz, University of Denver
- 27 **Teaching Mathematics for Social Justice: Examining Preservice Teachers' Conceptions**
Cindy Jong, University of Kentucky
Christa Jackson, Iowa State University
- 35 **Fostering Justification: A Case Study of Preservice Teachers, Proof-Related Tasks, and Manipulatives**
Jonathan D. Bostic, Bowling Green State University
- 45 **Playing with Mathematics: How Play Supports Learning and the Common Core State Standards**
Jennifer Mary Zosh, Pennsylvania State University, Brandywine; Brenna Hassinger-Das, Temple University; Tamara Spiewak Toub, Temple University; Kathy Hirsh-Pasek, Temple University; Roberta Golinkoff, University of Delaware
- 51 **Assessment of a Problem Posing Task in a Jamaican Grade Four Mathematics Classroom**
Kayan Lloyd Munroe, Hiroshima University

Fostering Justification: A Case Study of Preservice Teachers, Proof-Related Tasks, and Manipulatives

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ABSTRACT Instruction promoting reasoning and proof should be a part of K-12 mathematics teaching. Preservice teachers (PSTs) need mathematics content instruction that helps them construct valid justifications for mathematical notions and develop positive perceptions of proof. This case study describes instruction with proof-related tasks and manipulatives for middle-grades PSTs that draws on developmentally appropriate modes of argument representation. The purpose of the instruction was for PSTs to explore mathematics content through a lens of how a middle-grades teacher might use manipulatives to engage students in justifying mathematical statements in proof-related tasks. Results indicate that this instruction supported PSTs to justify mathematical ideas and that PSTs characterized their experiences positively. PSTs perceived proof in ways echoed in past literature, saw connections between the PSTs' instruction and future middle-grades classroom instruction they might enact, and identified the role of struggle while engaging in proof-related tasks. This case study provides ideas for mathematics content instructors to develop PSTs' understanding of mathematics through instruction that prepares them to engage their future K-12 students in proof-related tasks.

KEYWORDS *proof, justification, representation, preservice teachers, teacher education*

Instructional programs in mathematics should "enable all students to recognize reasoning and proof as fundamental aspects of mathematics" (National Council of Teachers of Mathematics [NCTM], 2000, p. 56). K-12 students are expected to build proficiency with increasingly sophisticated reasoning and proof techniques as they develop mathematical proficiency (Harel & Sowder, 2007; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). They need mathematical experiences that involve reasoning, justification, and proof in developmentally appropriate ways (Bieda, 2010; Stylianides, 2007). To meet that need, preservice teachers (PSTs) should have rich experiences learning mathematics at the college level that support them to frame their future mathematics instruction around fundamental aspects of mathe-

tics. The purpose of this manuscript is to describe, analyze, and present findings from a case study of middle school PSTs' experiences with justification through proof-related tasks and manipulatives.

Literature Review

Proof and Justification

Proof is a means to convince ourselves and others that a statement is true (or not true) and represents a form of understanding (deVilliers, 1990; Schoenfeld, 1994). Stylianides (2007) defines proof as

a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

1. It uses statements adopted by the classroom community (set of accepted statements) that are true and available without further justification;
2. It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and
3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 291)

This definition addresses representations as means to express validity of an idea statement (or lack thereof). This definition also includes sociocultural elements for proof, which are needed when examining research situated in classrooms. Teachers and students work collaboratively to justify statements and consider the potential validity of one's justification. In this frame, students develop knowledge of mathematics through convincing others of their assertions rather than assuming truth because (a) it is found in the textbook or (b) the teacher stated it was so. Thus, students and teachers can learn to engage in justification, a key component of learning to engage in proof (Bieda, 2010). Justification is the act of convincing someone that a statement is valid (Bieda, 2010). Proof extends justification in such a way that the mathematics community might accept the validity and truthfulness of these statements (Stylianides, 2007).

Bieda's (2010) case study of inservice teachers (ISTs) enacting proof-related tasks in middle school classrooms helps ground the present study. During one year, the ISTs in Bieda's study engaged in curricular professional development involving explorations and discussions of proof-related tasks, especially ones that focused on justification and proof, and reflected on ways to scaffold ISTs' students' learning during these tasks. Proof-related tasks engaged the students in developing conjectures and making generalizations (Bieda, 2010). One result highlighted by her analysis of ISTs' students learning from teachers in this professional development was that they offered conjectures to proof-related tasks on a regular basis and that students, in turn, shared justifications for those conjectures for approximately half the time, resulting in *proving events*. A proving event occurred when a conjecture and justification were offered. ISTs' reflections on these proving events indicated that justification "was seen as something 'especially impressive' or something for students who are developmentally ready" (Bieda, 2010, p. 380). A logical conclusion from Bieda's

work is that middle grades ISTs (and likely PSTs) need opportunities during professional development or teacher education coursework to build conceptions of justification as essential to fostering proving events in the classroom (Bieda, 2010). Not only should ISTs and PSTs enact tasks that foster proving events, but they should also hold positive perceptions of justification so that they can enact proof-related tasks and facilitate proving events as part of their instruction.

Knuth (2002) surveyed secondary ISTs about their perceptions of proof. Findings indicated that secondary ISTs believed proof could verify the truth of a statement, explain why a statement is true, communicate mathematical ideas, and demonstrate elements of mathematical structure. Broadly speaking, ISTs largely did not perceive that proof was important for their mathematics learning or their secondary students. Middle school ISTs and PSTs typically have less formal mathematics coursework than secondary ISTs and PSTs; therefore, it seems plausible that middle school teachers might also perceive proof in the same way as the secondary ISTs did in Knuth's study.

Modes of Representations

Mathematics involves numerous representations including symbols, graphs, and physical models (Goldin & Kaput, 1996). Goldin and Kaput's (1996) framework categorizes representations as *symbolic* or *nonsymbolic*. Concrete materials (i.e., mathematical manipulatives) and iconic representations, such as figures and diagrams, are considered nonsymbolic. Mathematical expressions and equations that involve variables, numbers, and constants are symbolic in nature. Such a framework can be used to examine modes of representation within proof and justification tasks (Bostic & Pape, 2010; Stylianides, 2007; Yee & Bostic, 2014). Representations are a vehicle for expressing one's ideas, and symbolic forms are not necessarily the best representation in every situation for students who want to communicate ideas in a coherent fashion (Stylianides, 2007; Yee & Bostic, 2014). Justification for ideas can arise from a picture, a unique organization of concrete tools, a series of symbolic expressions, or a combination of representations (Yee & Bostic, 2014). There are numerous concrete models that K-12 mathematics students use while reasoning about mathematics, which are collectively referred to as mathematical manipulatives. Mathematics instruction that encourages K-16 students to think with manipulatives has a positive impact on their ability to solve problems and justify mathematical statements (Carbonneau, Marley, & Selig,

2013). Mathematics teacher educators' use of manipulatives during proof-related instruction may be a means to foster sense making and justification of key mathematical ideas that PSTs might teach in the future.

Synthesis

PSTs need opportunities to engage in rich tasks with one another in a supportive learning environment focused on learning about mathematics and mathematical behaviors (NCTM, 2014). They should be adequately prepared to support their future students to justify mathematical statements in developmentally appropriate ways (Reid & Zack, 2009). Mathematics teacher educators are in the esteemed position where they can model such instruction to their students. The purpose of this study was to examine instruction using proof-related tasks and manipulatives in a content course for middle school PSTs (i.e., grades 4-9). The instructional aim was for them to explore mathematics content through a lens of how a middle-grades PST might use manipulatives to engage students in justification about proof-related tasks. There were two research questions for this study:

1. How successful are PSTs at producing justifications for topics arising from the middle school standards after experiencing proof-related instruction involving manipulatives?
2. How do PSTs perceive proof-related instruction involving manipulatives?

Method

Research Methodology

This study draws on the case study approach (Yin, 2003). In this study, the case is bounded by PSTs' experiences within instruction that used proof-related tasks and manipulatives as well as their outcomes during one undergraduate course. To maintain validity with this methodological approach, a description of proof-related instruction using mathematical manipulatives is shared to help the reader understand what PSTs experienced, followed by a report of outcomes resulting from this instruction.

Setting and Participants

The study took place at a midwestern university in which middle school PSTs completed a mathematics course during their junior year. The course is titled Mathematics Instruction for the Middle Childhood Teacher and

serves as a prerequisite for PSTs' middle grades mathematics methods course. The course's aim is to connect mathematics content and mathematics pedagogy. The previous 15 hours of undergraduate mathematics prior to this course included 6 hours of calculus and 3 hours each of statistics, geometry, and algebra designed for PSTs. Mathematics department faculty taught these sophomore-level courses. Mathematics instructors for the previous courses confirmed that PSTs had seen proofs drawing on symbolic representations prior to this course in other content courses, but had not seen proofs involving nonsymbolic representations. Mathematics instructors had no expectations for these PSTs to prove ideas, much less justify arguments as valid. There were 12 female and 4 male PSTs enrolled in this course. All PSTs' names are pseudonyms.

Proof-Related Instruction

Setting the stage. Prior to instruction, the instructor formatively assessed PSTs' knowledge of middle grades mathematics content through formal and informal conversations. Discussions about mathematics content involved topics such as divisibility rules, prime and composite numbers, and operations with rational numbers, all of which are addressed in the fourth-grade Standards for Mathematics Content (NGA & CCSSO, 2010). Conversations with PSTs led to an impression that they could state and perform procedures but could not justify their validity. Tasks were developed through reviewing the Standards for Mathematics Content (see NGA & CCSSO, 2010) for key ideas, drawing on conversations with past middle childhood mathematics instructors, and examining tasks found in mathematics textbooks for preservice and inservice elementary and secondary mathematics teachers. The mathematical topics for these tasks included divisibility rules, prime and composite numbers, and operations with rational numbers.

Proof-related instruction in this study was framed as the interplay between proof-related tasks, discourse, and learning environment. PSTs were expected to justify statements and conjectures as well as develop valid arguments for why a statement or conjecture was true or false. Proof-related instruction lasted 8 consecutive weeks during the 16-week semester. Sessions were held twice each week for 75 minutes. PSTs sat in groups of 4 or 5 at 4 tables throughout the room. Boxes of various manipulatives (e.g., bi-color counters, snap cubes, pattern blocks, and base-ten blocks), blank lined and unlined paper, and colored pencils were placed at every table.

Week 1. The first week of instruction involved characterizing proof and justification as well as discussing representations and mathematical terminology found in the middle grades. Another building block was focusing PSTs' attention on being precise with their mathematical language, much like the expectations discussed in the sixth Standard for Mathematical Practice (SMP): *Attend to precision* (NGA & CCSSO, 2010). The instructor modeled this practice for PSTs and encouraged PSTs to reflect on their language use as well as others' during class meetings. Specifically, PSTs gave constructive feedback to their peers when an imprecise word or phrase was used (e.g., "a divides b with no remainder" instead of "a divides b evenly"). Similarly, instruction was designed to engage PSTs in SMP 3: *Construct viable arguments and critique the reasoning of others* (NGA & CCSSO, 2010). SMP 3 indicates that

mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. . . . They justify their conclusions, communicate them to others, and respond to the arguments of others. . . . Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. (p. 6-7)

These and other SMPs provided the background for what might be valued during the intervention: (a) using assumptions and definitions, (b) discussing reasoning with others, (c) stating the meaning of the chosen referents, and (d) reflecting on ways to link formulated explanations in elementary grades "to examine claims and make explicit use of definitions" (NGA & CCSSO, 2010, p. 7).

Weeks 2–8. The other 7 weeks were spent examining mathematics found in the middle grades mathematics standards (NGA & CCSSO, 2010). In total, PSTs investigated 7 mathematical statements such as "If a number is divisible by both two and three then it must be divisible by six." A full schedule of the mathematical statements covered in Weeks 2–8 is provided in Appendix A. PSTs were presented with a statement to justify at the beginning of class. Then, they worked individually for 10 minutes and were encouraged to use any mode of argument

representation they felt was appropriate to express their reasoning. At first, PSTs were hesitant to take manipulatives from the boxes. After some encouragement from the instructor to think about representations they might use to express their ideas, they more readily took them the following week. By Week 3, students considered and selected manipulatives to use during justification tasks presented in class.

After a period of time for independent thinking, PSTs were encouraged to collaborate with a partner for approximately 45 minutes to craft justification of their ideas that was mathematically correct and appropriate for a middle-school child. The instructor circulated during this small-group time and asked probing questions to scaffold PSTs' thinking (e.g., "What does this [a figure, symbol, or manipulative] represent?"). PSTs' peers also asked questions (e.g., "Can you show me what you mean in a different way?"). Later, groups presented their justifications during a whole-class discussion, which lasted approximately 15 minutes. PSTs were encouraged to ask questions to their peers when an idea was unclear. The instructor did not ratify the PSTs' justifications. Instead, he asked PSTs to reflect on two guiding questions as a means to assess the quality of their justification:

- a. Is the justification and all components of it mathematically valid?
- b. Is there a good possibility that a middle grades student might understand the justification?

As a result of these conversations, PSTs ratified each justification as valid or invalid and gave feedback about ways to strengthen it using these guiding questions.

Data Sources

Two data sources were used for investigating student outcomes. The first was a think-aloud interview conducted as part of PSTs' final exam. PSTs came to the instructor's office near the end of the semester and were asked to justify two mathematical statements stemming from the middle grades mathematics curriculum. The statements were different from ones seen during instruction but drew on related topics. PSTs had access to various mathematics manipulatives (e.g., base-ten blocks, pattern blocks, and bi-color counters), blank paper, and markers. PSTs were told that they may engage in the tasks using any means necessary; manipulatives were available but not required to complete the tasks. The instructor did not ratify PSTs' questions. Interviews lasted approximately 10-25 minutes per PST and were videotaped. First, PSTs were given a warm-up question that

read “Justify that the sum of any two even numbers is always an even number.” After answering this question, the instructor offered the first target task: “Justify that every integer greater than one is either prime or can be expressed as a product of primes.” When PSTs indicated that they were done, they returned the first target task and were given the second one. The second target task stated “Justify that any proper non-unit fraction can be decomposed into a sum of fractions with the same denominator.”

The second data source was a survey administered near the end of the semester adapted from Knuth’s (2002) survey of secondary ISTs’ perceptions of proof. The instrument (see Appendix B) asked PSTs to reflect on their recent instruction with proof-related tasks and indicate to what degree the instruction impacted their perceptions of such instruction for the middle grades. PSTs completed the survey outside of class.

Data Analysis

PSTs’ think-aloud interviews were analyzed in two ways. First, responses were coded for validity. A valid justification drew upon sound reasoning, started with a known premise, and logically connected each premise to a final conclusion (i.e., the idea that was to be justified). A score of zero (no credit) meant that an argument was invalid. A score of one (partial credit) indicated that the justification had some appropriate features but was incomplete or not entirely valid. A score of two (full credit) meant that the justification was valid, complete, and had sound reasoning. Next, justifications were coded for the mode of argument representation that the PSTs used to convey ideas. Representations were coded as Symbolic or Nonsymbolic using Goldin and Kaput’s (1996) framework.

Survey responses were examined for themes within and across questions (Hatch, 2002). First, the responses for each question were gathered across all participants. Next, the responses were read, and initial impressions about PSTs’ perceptions of proof and instruction with proof-related tasks were drawn. Then, the responses were examined again with these initial impressions in mind. Those initial impressions, along with a preponderance of evidence and paucity of counterevidence, later became themes. Every attempt was made to limit the number of themes to those that were most significant across all participants.

Findings

PSTs’ Performance and Representations

Results related to the validity of PSTs’ justifications as well as their modes of argument representations employed during the think-aloud interview are shared in Tables 1 and 2.

Table 1

Frequency of PSTs’ Success at Completing Target Task Proofs

Score	Target Task 1	Target Task 2
Valid	14 (87.5%)	14 (87.5%)
Partially Valid	1 (6.25%)	1 (6.25%)
Invalid or Not Attempted	1 (6.25%)	1 (6.25%)

Table 2

Frequency Indicating PSTs’ Modes of Argument Representation

Representation Mode	Target Task 1	Target Task 2 ^a
Symbolic	1 (6.25%)	2 (13.3%)
Nonsymbolic	15 (93.75%)	13 (86.7%)

^a Note that 1 PST did not attempt Task 2.

PSTs were successful at being able to express mathematically valid justifications. Every PST was able to justify one of the prompts, and most justified their ideas appropriately. Nearly everyone used manipulatives during the think-aloud interview; no PST used pictures or drawings for either target task. All 15 PSTs who used nonsymbolic representations (i.e., manipulatives) for the first target task provided valid or partially valid justification. Twelve of the 13 PSTs who drew upon nonsymbolic representations for the second task offered valid justification. The frequencies indicate that PSTs’ use of manipulatives provided a means to represent their thinking while working to justify the statement.

A portion of Rory’s case is shared to describe a typical think-aloud for the sample. Broadly speaking, interviews had three stages. The first stage involved exploring a set of connected examples. Initially, Rory engaged in the first task by examining composite numbers 15 and 24:

"I'm thinking of prime factorization; so, I'm going to start with looking at 15 because it's odd...now I want to look at 24 because it's even." She represented each value using snap cubes in an array fashion and then sought to make sets of blocks, each representing a prime number (e.g., three blocks meant the number three). The second stage involved examining a different set of examples from the first set, which for this task was a prime number. "I can't break three into a smaller prime number; so, I know it must be prime. I'm going to work on that...I'm going to try 13 and see if the same thing happens." During this stage, she grabbed 3 snapcubes and described how this example was a prime number. "I addressed the other part of the statement [Justify that every integer greater than one is either prime or can be expressed as a product of primes.]" The third stage involved abstracting away from earlier work during the first and second stages of the task. During the third stage of the first task, Rory explored an unknown quantity, representing it as a variable (see Figure 1). "So I'm going to construct a number (grabs snap cubes) but I don't know what it is (connects snap cubes). Let's call it 'x'...I could count them but that's not the point. This is some number 'x'."

Rory clarified that her purpose during this stage was to construct an unknown quantity much like a variable during work with symbolic representations. She proceeded to describe how she could decompose the unknown quantity into sets of prime numbers in much the same way as 15 or 24 (her previous cases) but if those attempts were unsuccessful then the unknown quantity must be prime. At one point she shared "If I can make sets of three, just like I did for 15 and 24, then I can decompose the unknown number. But, if I can't, then either the number is prime, like 3 was, or it's divisible by other factors." Rory's use of snap cubes was evidence that she could represent an abstract idea (i.e., number) in a concrete fashion, which is evidence of her capability



Figure 1. Rory's representation of an unknown quantity.

to translate between representations. Relatedly, she aimed to make generalizations and employed nonsymbolic representations to express her ideas. Rory's case was similar to that of her peers and highlighted the type of thinking PSTs used during the think-aloud interviews.

PSTs' Perceptions of Proof and Reasoning

Three themes emerged from PSTs' survey responses about their experiences stemming from this instruction. First, PSTs believed that proof was about convincing oneself and others whether a statement was true and drawing upon relationships between concepts. Suzy commented "proof allows students to understand why a 'rule' or statement is true. It allows them to gain a deeper understanding of why things are true." Janet suggested that proof is "showing why something ends up being the way it is. So, demonstrating why the math [concepts and procedures] works the way it does." These statements, among numerous others, painted a clear picture that proof meant helping students to understand why mathematics works and makes sense.

A second theme was that PSTs perceived similarities between their experiences with proof-related tasks as well as manipulatives and teaching middle-grades children through inquiry-based, student-centered approaches: both promote a deep understanding of mathematics. Anne commented "proof in the middle grades is special because it needs to be hands-on and done in multiple ways for students with different learning styles." Anne also felt that "the teacher and learning environment must be welcoming and enjoyable for students." Others suggested that middle-grades proof-related instruction required a safe, caring learning environment with appropriate norms and a teacher that fostered peer-to-peer discourse. Marie wrote, "We worked in groups to come up with justifications.... This helped us share ideas. It also helped when groups showed their justifications to the whole class because we saw them in different ways and from different perspectives." Her comments about working in groups, sharing ideas within groups and with the whole-class, and considering others' ideas are connected to notions of learning environment as described by NCTM (2007). PSTs indicated that small-group and whole-class discussions about justifications as well as critically examining peers' language used to explain these statements supported them to generate ideas for their future middle-grades instruction.

Finally, there was a common theme of developmentally appropriate struggle and building from one's prior knowledge using manipulatives as a mode of argument representation. Rob said, "The big aspect was allowing

us to struggle with the ideas done in class. The struggle caused me to think further, and using manipulatives helped give me another view.” This other view that Rob mentioned during the instruction allowed Rob to make connections between mathematical statements and conjectures so that he might be able to justify his work to himself and his peers. Thus, encouraging manipulatives as a mode of argument representation (i.e., forms of expression) supported Rob to persevere while struggling to justify a challenging mathematical statement. Chelsea echoed Rob’s sentiment: “I have always struggled understanding proofs and being able to justify my ideas. I have a little more confidence in justifying my ideas. I like to use manipulatives to help explain it [argument] and feel more confident in showing specific cases.” These PSTs indicated that the rigor of the justification should match the students’ abilities and be within their cognitive grasp. For instance, Lynn commented, “The teacher cannot expect students to automatically understand proofs, much less justifications for some theorem or property. It is a process and the teacher must guide students to understand why something is true.” Lynn’s comments remind mathematics teacher educators of the importance of perceiving PSTs’ growth as a process that continues beyond the end of one course and must be considered within the constellation of a mathematics teacher education program

Conclusions

The purpose of this study was to explore instruction with proof-related tasks and manipulatives. There are two key findings from this study. First, PSTs successfully justified their arguments using nonsymbolic modes of argument representations (i.e., manipulatives). Second, evidence indicated that using manipulatives during instruction with proof-related tasks was associated with positive perceptions of proof-related instruction. Taken collectively, these results suggest that (a) these middle school PSTs were able to justify ideas they might teach in the future and (b) the instruction promoted positive conceptions about the role of proof and justification in their future classrooms. Conclusions from the present study are connected to Stylianides’ (2007) framework as well as the two focal studies (i.e., Bieda, 2010; Knuth, 2002).

The definition of proof for the present study includes a sociocultural element, and PSTs shared that this element was quite important. PSTs’ words about the community’s (i.e., other PSTs’) role when sharing statements,

suggestions, and feedback on arguments suggests they valued working collaboratively to justify statements and develop arguments. Anne and Marie’s comments indicate how working to develop an argument in collaboration with others in the classroom community helped them. Furthermore, Lynn’s comment about the role of the teacher addresses Stylianides’ (2007) position that the teacher’s role during proof-related instruction is (a) to be a steward of the discipline and (b) to foster connections across mathematical topics. Thus, this study highlights the role and importance of the sociocultural nature of proof-related instruction. Both Bieda and Knuth urge the mathematics education field to explore PSTs’ perceptions of proof and enact instruction with proof-related tasks at the undergraduate level so PSTs may be more likely to engage their future middle school students in proving events. The findings from this case study suggest that instruction with proof-related tasks and manipulatives during undergraduate mathematics coursework has the potential to foster positive perceptions of proof-related instruction. These findings imply that middle school PSTs from this study are interested in implementing proof-related tasks in their future middle school classrooms.

Limitations

The inception of the project spawned after an initial discussion at the beginning of the semester with PSTs and mathematics content faculty. The results of this study are held tentatively due to its posttest-only design. A second limitation is the nature of the mathematics explored in this study. Similar instruction with proof-related tasks and manipulatives focusing on algebraic or geometric concepts would assuredly involve different manipulatives.

Implications for Preparing Mathematics PSTs

First, this instructional approach should supplement, not replace, current mathematics content instruction aiming to support future PSTs’ ideas about proof and justification. The proof-related instruction with manipulatives was done during half of the semester. The other time was devoted to exploring mathematics concepts, not necessarily engaging students in justification. PSTs seemed to develop appropriate perceptions about proof through proof-related instruction that used manipulatives. Furthermore, initial student teaching experiences in middle-grades classrooms where proof and justification are not valued may prevent PSTs from enacting instruction with proof-related tasks. Thus, the findings about proof-related instruction are held tentatively. Such instruction,

as described in this manuscript, may be a means to foster appropriate perceptions about proof and justification.

A second implication from this study is that mathematics content instruction for PSTs that adheres to best practices (NCTM, 2000, 2007, 2009, 2014) is no different from instruction with proof-related tasks and manipulatives enacted in this study. PSTs and other mathematics instructors converged on the idea that PSTs needed opportunities to provide some justification for why something is true (or not true). Proof-related instruction during university coursework may support middle-grades PST future middle school students to engage in proof and justification. PSTs needed time and instructional opportunities to struggle with mathematics, just like they hope for their future students.

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Appendix A

Mathematical Statements Explored During Instruction

Week 2

The sum of an even and odd integer is always odd.

Week 3

The sum of two odd integers is always even.

Week 4

The product of two odd integers is odd.

Week 5

The product of an even and odd integer is always even.

Week 6

The divisibility rule for 2: If an integer greater than or equal to 2 has a unit digit divisible by 2 then the integer must also be divisible by 2.

Week 7

The divisibility rule for 3: If an integer greater than one has digits that sum to a value divisible by three then the integer must be divisible by three.

Week 8

The divisibility rule for 6: If a number is divisible by both two and three then it must be divisible by six.

Appendix B

Survey

Directions: Please respond to the following prompts. Feel free to use the back of the paper if you need more room.

1. What does the notion “proof” mean to you?
2. What constitutes proof in the middle grades?
3. What is the difference between an explanation and justification?
4. What aspects of middle grades learning environments promote learning about proof and reasoning?
5. What are some essential aspects of tasks that promote proof and reasoning in the middle grades classroom?
6. What aspects of Mathematics Instruction for the Middle Childhood Teacher instruction prepared you to teach proof and reasoning to middle school students? Feel free to share examples.
7. What aspects of Mathematics Instruction for the Middle Childhood Teacher instruction did not prepare you to teach proof and reasoning to middle school students? Feel free to share examples.
8. Have you experienced any changes in your confidence to do proofs as a result of Mathematics Instruction for the Middle Childhood Teacher instructional experiences? If so, please describe these changes.
9. Have you experienced any changes in the ways that you complete proofs as a result of Mathematics Instruction for the Middle Childhood Teacher instructional experiences? If so, please describe these changes.
10. Have you experienced any changes in your perceptions about ways to carry out proofs as a result of Mathematics Instruction for the Middle Childhood Teacher instructional experiences? If so, please describe these changes.