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THINKING IN PATTERNS TO SOLVE MULTIPLICATION, DIVISION, AND FRACTION PROBLEMS IN SECOND GRADE

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ABSTRACT Experts think in patterns and structures using the specific "language" of their domains. For mathematicians, these patterns and structures are represented by numbers, symbols and their relationships (Stokes, 2014a). To determine whether elementary students in the United States could learn to think in mathematical patterns to solve addition and subtraction, a pilot curriculum using an Asian model of base-10 counting was introduced in kindergarten and first grade. Children in the pilot program, like their Asian counterparts, did not appear to have any difficulty with the concept of place value. They also did not have significant difficulties with single- and double-digit addition and subtraction as is often the case with American students (Stokes, 2010, 2013, 2014a, 2014b). To continue the pattern-based process in second grade, a Multi-Operation Chart was designed to make the relationships between multiplication, division and fractions clear. Testing at the end of the school year showed that students not only met, but exceeded the Standards set by the Common Core standards for second grade. Medians and modes for correct single- and double-digit multiplication, single-digit division and fractions were 100%. Educational implications are offered and discussed.

KEYWORDS *mathematical development, second grade, decomposition, multiplication, division, fractions, multi-operation chart, numeric-symbolic patterns, place value, Common Core*

If we ask "how does a mathematician think about mathematics," the answer will usually be in patterns. The thinking is done in the language of the domain—numbers and symbols. In considering a related question, "can a child learn to think in patterns, like a mathematician?" we can look at one of the earliest patterns that children encounter in mathematics: single-digit addition. Our answer should be "no" if the child is applying a set of rules acquired by rote; it should be "yes" only if she, like the expert mathematician, has learned to recognize the numeric-symbolic patterns inherent in addition and, as a result, can apply them to her thinking and problem solving.

What types of patterns? The most basic one is that numbers are combinations of other numbers. More specific patterns involve pairs of numbers, for example, $4 + 1 = 5$ and $3 + 2 = 5$ are combinations that both give us the number 5. Another pattern, the commutative property of addition, is that order of the numbers does not affect the result: $3 + 2 = 2 + 3 = 5$.

The pilot curriculum presented in this paper was designed to teach children how to think in numeric-symbolic patterns. The theoretical background and development of the program are derived not from mathematics education per se, but from broader consideration of expertise and problem solving in general.

Expertise

Expertise has been defined as a continuum, "a sequence of mastered challenges with increasing levels of difficulty in specific [domains]" (Zimmerman, 2006, p.706). Novices are found at the low end of the continuum, experts are found at the high end. The difference between the two is that experts not only know more, their knowledge is "organized in ways that facilitate effective problem solving" (Nokes, Schunn, and Chi, 2010, p. 269).

Expertise and patterns

Experts perceive and problem-solve using large meaningful patterns in the areas and in the languages of their expertise (Chi, 2011; Chi, Glaser, & Farr, 1972). For mathematicians, the patterns are represented by numbers and symbols. For chess masters, they are represented by legitimate arrangements, i.e., game positions reached by correct moves of the chess pieces (Chase & Simon, 1973); for expert programmers, by clusters of subroutines (Soloway, Adelson, & Ehrlich, 1988).[1]

Experts also readily access the patterns stored in long term memory as highly integrated associative networks. Ready access, which relies on retrieval cues linked to relevant networks, results in computational efficiency, allowing the expert to bypass the limited capacity of short-term or working memory. This ability, called skilled memory or Long-Term Working Memory (Ericsson & Kintsch, 1995), accounts for the expert's ability to temporarily store extensive task-relevant materials.

Finally, experts are more effective than novices in elaborating and expanding already acquired patterns, which makes it easier for them to acquire new domain-relevant information (Nokes, Schunn, & Chi, 2010). This is even seen in young children. For example, compared to children who use fewer strategies, those who use more strategies (an indication of developing expertise) when solving addition or subtraction problems acquire new strategies (and develop their associative networks) faster (Siegler & Jenkins, 1989).

Expertise and deliberate practice

In established domains, relevant patterns are acquired in well-established sequences in a process called "deliberate practice" (Ericsson, 2006). Deliberate practice is focused, continuous and variable, involving a well-defined task (like addition) with an appropriate level of difficulty (single or double digit) for the learner (a first grader), evaluation (by the teacher) and assignment of additional, specific problems to correct errors or misconceptions and to extend students' understanding of the task.

Expertise and early mathematics curriculum

The question that prompted the pilot curriculum was straightforward: with practice, and practice primarily in numbers and symbols, could novices (i.e., American children, K-2) learn to think and problem solve like experts (i.e., mathematicians) in numeric-symbolic patterns?

Introducing the Pilot Curriculum (K-1)

Assuming that the neuro-anatomy of the brains of Asian and American children do not differ, differences in their mathematical performance (OECD, 2012; Stevenson, Chen, & Lee, 1993) could involve pattern-based abilities. Place-value, which is a core problem for American but not for Asian children (Fuson, 1990), was a logical place to look for such a pattern.

The Place Value Problem

The term place-value simply means that the placement of each digit in a multi-digit number determines the number's value. In a double-digit number, the values of the two places are tens (represented by the digit on the left) and ones (represented by the digit on the right). The problem is basically this: given the number 14, American children mistake the digit on the right, which represents 4 ones, as being of greater value than the digit on the left, which represents 1 ten. Asian children who call the same number "ten-four" do not make the same mistake (Miura & Okamoto, 2003). Critically, the Asian count makes the patterned base-10 structure of the number system explicit; the American count obscures it. Since mastery of place-value facilitates mastery of addition and subtraction (Fuson & Kwon, 1992) an explicit base-10 count was identified as a critical pattern to be acquired.

The K-1 Curriculum

The curriculum, called *Only the NUMBERS count©*, is based on three substitutions: an explicit base-10 count replaced the English language count; a single manipulative (the Count-and-Combine Chart) replaced the multiple arrays seen in American classrooms; an exclusive focus on manipulating numbers, symbols, and patterns replaced the wordiness of typical curricula.

[1] The expert perception-performance connection is the product of procedural knowledge and a resulting "habit of noticing and acting on noticing" (Simon, 1988). The basic mechanism is operant conditioning. The format of a perception-action operant is simple: if the situation is X, then do Y. X is the perception, Y is the action.

The Count

Table 1 presents an abbreviated example of the count in English. There are several things to notice. First, the first ten number names (1 to 10) combine in logical, iterative patterns to form higher numbers. Second, the number name ten appears in every number above ten (e.g., 11 is *ten*-one, 21 is two-*ten*-one). Third, ten is treated as a unit. It is not 10 ones, it is itself, one ten. The count makes the patterned structure of our shared base-10 number system explicit.

The Manipulative

Figure 1 shows three rows of the first Count-and-Combine Chart (1 to 10), all parts of which were movable. Children recited the rows in order: "Number 1 same as word 1 equals 1 block, Number 2 same as word 2 equals 2 blocks……" They arranged and re-arranged the blocks (laminated squares) to create addition combinations. The bottom line shows one combination for the number 3.[2]

Figure 2 shows the row representing 13 (ten-three). Notice that ten is represented by a single block marked "10," and that the combination for 13 mirrors that for 3 in Figure 1. Children also decomposed the 10, for example: 5 + 5 + 2 + 1 = 13.

Earlier studies showed that children taught with *Only the NUMBERS count*© in both kindergarten and first grade were proficient in place value, as well as single and double digit addition and subtraction, outperforming matched comparison children taught with the district curriculum (Stokes, 2014a; 2014b). This suggested that similarly prepared students might be ready for multiplication, division, and fractions in second rather than in third grade. This paper presents that expansion.

Table 1

Explicit base-10 count in English. The first column shows the ones; the second, the tens. The third shows how the numbers recombine to form twenties, thirties, etc.

Ones	Tens	Twenties, etc.
	10 *ten*	20 two-*ten*
1 one	11 *ten*-one	21 two-*ten*-one
2 two	12 *ten*-two	22 two-*ten*-two
3 three	13 *ten*-three	………………
4 four	14 *ten*-four	30 three-*ten*
5 five	15 *ten*-five	31 three-*ten*-one
6 six	16 *ten*-six	………………
7 seven	17 *ten*-seven	40 four-*ten*
8 eight	18 *ten*-eight	41 four-*ten*-one
9 nine	19 *ten*-nine	………………

Expanding the Pilot Curriculum: 2nd Grade

The major innovation was integrating the teaching of multiplication, division, and fractions. Like the K-1 curriculum, the expansion was based on substitutions. There were three.

1. The Multi-Operation Chart (adapted from the Chinese) replaced the standard multiplication table. The Chinese table goes up to 9 and is only used for multiplication. The new chart goes up to 10 and is used for multiplication, division, and fractions.
2. Reciting the chart as multiplication, division, and fractions replaced reciting the standard multiplication table as addition.

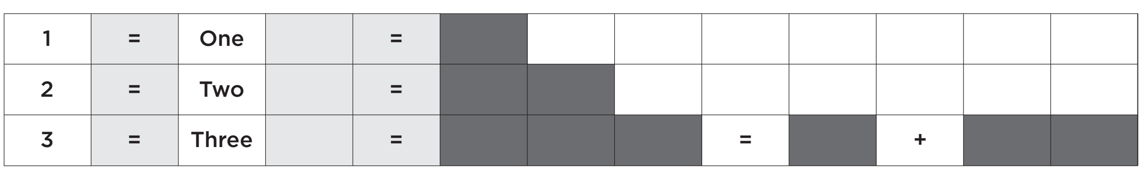

Figure 1. Count-and-Combine Chart 1 to 3.

Figure 2. Count-and-Combine row for 13 (ten-three).

[2] There are two others: 2 + 1 and 1 + 1 + 1. While these are technically permutations (which take order into account), the term "combinations" (in which order does not count) was used to reinforce the idea that numbers are combinations of other numbers.

3. Teaching multiplication, division, and fractions as interconnected patterns, replaced teaching the three separately.

Method

Participants

Participants in the study, students who had been taught with *Only the NUMBERS Count©* during kindergarten, first, and second grade, were in three different second grade classes at a suburban public school. Classes were sorted by gender (to balance the number of girls and boys) and not by ability. The school adhered to the Core Curriculum State Standards (2011) in mathematics. The new materials replaced materials from the district curriculum *(enVisionMATH)* covering numbers and numeric relations. The district curriculum was used for topics (e.g., graphs, measurements) outside the scope of the new curriculum. Daily mathematics lessons lasted for an hour. Teachers were required to use *Only the NUMBERS Count©* for a minimum of fifteen minutes.

Students

There were 8 female and 16 male students. Of these, 8 were Hispanic, 8 were White, 3 were African-American, 3 were multi-racial, and 2 were Asian. Nine students qualified for free or reduced fee lunch. All were proficient in English and 6 were bi-lingual, speaking a language other than English at home. Mean age at the time of post-testing was 8 years, 2 months (98 months); range was 7 years, 8 months (92 months) to 8 years, 7 months (103 months).

Teachers

Two classes had the same teachers for the entire school year. One teacher had four years' experience teaching second grade; the other had nine years' experience teaching both second and third grades. The third class had two teachers. The first, who went on leave at the end of December, taught for 23 years. She had experience with kindergarten, first, second and fifth grade classes. The second was a substitute who taught at the school for three years. She was the only teacher with prior experience using the intervention (in kindergarten and first grade). Professional development was provided by the researcher.

Materials

The Multi-Operation Chart

The chart is shown in Figure 3. There are several things to notice about the format. First, the upper left cell contains symbols for multiplication and division. Second, each column begins with the square of its number. For example, the twos column begins with 4 (2 x 2 = 4). This is because 2 (1 x 2 = 2) is included in the ones column.

÷ ×	1	2	3	4	5	6	7	8	9	10
1	**1**									
2	2	**4**								
3	3	6	**9**							
4	4	8	12	**16**						
5	5	10	15	20	**25**					
6	6	12	18	24	30	**36**				
7	7	14	21	28	35	42	**49**			
8	8	16	24	32	40	48	56	**64**		
9	9	18	27	36	45	54	63	72	**81**	
10	10	20	30	40	50	60	70	80	90	**100**

Figure 3. Multi-Operation Chart used for multiplication, division, and fractions.

Third, given that each column begins with the square of its number, there are 45 empty cells. The empty cells help the children "see" the interconnected patterns (one of which is highlighted and explained below).

Unlike the multiplication table used in American classrooms, which children recite and learn as addition series (2, 4, 6, etc.), the columns and rows in the chart are initially recited as multiplication series. For example, the twos column is recited as "two 2s are 4, two 3s are 6, two 4s are 8, two 5s are 10…." It is also recited in reverse by rows. Starting (in the leftmost column) at the threes row and moving down, the recitation is "three 2s are 6, four 2s are 8, five 2s are 10…."

The use of the chart for division and fractions is what makes it unique. Once the children are familiar with a column, they are taught that—just as subtraction "un-does" addition—division "un-does" or reverses multiplication.[3] To see how the chart facilitates such understanding, look at the highlighted intersection of the 3 row and the 2 column. The inter-section captures the pattern: 2 x 3 = 6, 3 x 2 = 6 and 6 ÷ 2 = 3 and 6 ÷ 3 = 2. They are then taught that unitary fractions are a kind of division. Children learn that one half (1/2) means divide by 2, and one third (1/3) means divide by 3. Just as in division, the answers (and the recitation) are at the intersection: "1/2 of 6 is 3, 1/3 of 6 is 2." Children now practice reciting the entire pattern (multiplication, division, fractions) for a given intersection: "2 times 3 is 6, 3 times 2 is six, 6 divided by 3 is 2, 6 divided by 2 is 3, 1/2 of 6 is 3, 1/3 of 6 is 2."

The Blocks

Blocks (like those on the Count-and-Combine Chart) were arranged to represent multiplication and division. The number of rows indicate how many groups. The number of blocks in a row indicate how many in each group. Figure 4 shows the arrangement for the multiplication problem "two 3s are 6." It also represents the division problem "If 2 groups of the same size together equal six, how many are in each group?"

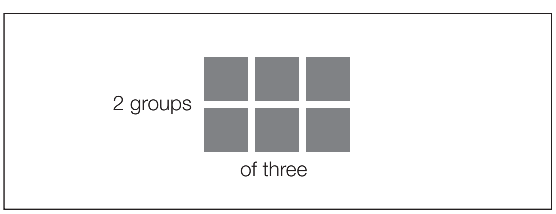

Figure 4. Example of block arrangement for multiplication and division.

In accordance with Simon's (1988) seminal work on learning by doing, students manipulated the blocks, used numbers and symbols to write problems, and recited the related multiplication, division, and fraction patterns every day.

Procedure

The study was conducted in three phases. Phase 1 involved pre-testing at the end of first grade to assess student readiness. Phase 2 involved weekly second grade classroom visits. Phase 3 involved post-testing at the end of second grade. Testing and observations were done by the researcher and four undergraduate research assistants.

Phase 1: Assessing Readiness. Table 2 lists items on the first grade test.

Table 2
Pre-Test Items for End of First Grade

Category	Content
Count to 100	Counting was coded as correct up to the first error (if a child counted 11, 12, 15, her score would be 12).
10s in 30 & 50	Children were asked how many tens there are in 30 and in 50.
Numbers/ symbols	Children read aloud numbers (1, 2, 3, 4, 5, 7, 8, 12, 15, 17, 20, 32) alone or combined with symbols (plus, minus, equals) in problems like 2 + 2 = 4.
Place value	Children were asked (a) to read aloud the written numbers 16, 25, 31, 56, 11; (b) to say which of each pair was bigger and (c) to explain their answers.
Addition	Children solved three problems with single addends (3 + 5, 6 + 6, 9 + 7) and three with double addends (10 + 18, 21 +11, 35 + 17).
Subtraction	Children solved three problems with at least one single digit (5 – 3, 10 – 5, 13 – 6) and three with double digits (20 – 20, 32 – 15, 25 – 19).
Combinations	Children made up two problems with two addends (_ + _ = 8) and one problem with three (_ + _ + _ = 8) for each of the following sums: 8, 10, 16, 25.
Word Problems	Children solved four addition and two subtraction word problems.

[3] "Un-do" is a simple, non-technical term that the children understood. In a higher grade, the more specific term "inverse" (i.e., mapping the solution back to the original input values) would be introduced.

Phase 2: Visiting the three second grade classrooms. During the school year, the researcher and four undergraduate lab assistants visited the school on Fridays, observing and assisting teachers and students. The teachers used a how-to-work book (Continuing *Only the NUMBERS count*©). At the start of the year, the researcher demonstrated how the chart and blocks were to be used. After this, regular classroom teachers did all the teaching.

Phase 3: Assessing New Knowledge. Table 3 shows the items in the post-test given to second graders at the end of the school year. Children were tested individually and asked to explain how each problem set was solved.

Table 3

Post-Test Items for End of Second Grade

Category	Content
Count by 10s	Coding was counted as correct to the first error.
Number of 10s	Children were asked how many tens there are in 30 and in 50.
Place	Children (a) read the written numbers 16, 25, 31, 56, 11 aloud; (b) said which of each pair was bigger and (c) explained their answers.
Addition	Children solved three problems with at least one single-digit addend (6 + 5, 16 + 5, 11 + 9) and two with double-digit addends (21 + 11, 21 + 19).
Subtraction	Children solved two problems with at least one single-digit subtrahend (10 – 5, 18 – 6) and two with double-digit subtrahends (20 – 20, 32 – 12).
Combinations	Children made up two problems with two addends (_ + _ = 8) and one problem with three (_ + _ + _ = 8) for each of the following sums: 8, 16, 32.
Multiplication	Children were given five single digit problems (2 x 6, 8 x 2, 3 x 5, 4 x 4, 7 x 5) and one problem with a double digit multiplicand (12 x 3).
Word Problems	Children solved six word problems requiring addition (in 4), subtraction (in 2) and multiplication (in 3).
Division	Children solved the following problems: 20 ÷ 2, 14 ÷ 7, 16 ÷ 4, 15 ÷ 3.
Fractions	Children solved the following problems: ½ of 20, ½ of 12, 1/3 of 15, ¼ of 16.

Results

First Grade Scores

Average test scores, calculated as percentage correct, are shown in Table 4. Means, standard deviations, medians and modes are shown. Not all means are 100%. However, since over half the students had perfect scores, all medians and modes are 100%. The results indicated that the students were prepared to learn multiplication, division, and unitary fractions.

Table 4

Post-Test Scores for First Grade (2013-2014)

Category	Mean	SD	Median	Mode
Count to 100	100.00	.00	100.00	100.00
by tens	99.55	2.13	100.00	100.00
Tens in 30 & 50	97.73	10.66	100.00	100.00
Number/Symbol	100.00	.00	100.00	100.0
Place Value	78.41	41.04	100.00	100.00
Addition				
Single	96.91`	10.00	100.00	100.00
Double	81.62	28.73	100.00	100.00
Subtraction				
Single	100.00	.00	100.00	100.00
Double	90.14	21.66	100.00	100.00
Combinations				
First	92.32	17.81	100.00	100.00
Second	90.77	18.57	100.00	100.00
Third	89.23	19.17	100.00	100.00
Word Problems				
Addition	92.05	19.49	100.00	100.00
Subtraction	73.81	37.48	100.00	100.00

Second Grade Scores

Average second grade scores, calculated as percentage correct, are shown in Table 5. Means, standard deviations, medians and modes for each measure are included. As with first grade scores, all medians and modes are 100%, while means reflect lower scores by less than half the children.

Table 5

Post-Test Scores for Second Grade (2014-2015)

Category	Mean	SD	Median	Mode
Count by tens	91.67	28.23	100.00	100.00
Tens in 30 & 50	100.00	.00	100.00	100.00
Place Value	92.71	24.98	100.00	100.00
Addition				
Single	97.21	13.68	100.00	100.00
Double	93.75	16.89	100.00	100.00
Subtraction				
Single	93.75	16.89	100.00	100.00
Double	100.00	.00	100.00	100.00
Combinations				
First	100.00	.00	100.00	100.00
Second	100.00	.00	100.00	100.00
Third	97.17	9.59	100.00	100.00
Multiplication				
Single	93.33	21.80	100.00	100.00
Double	79.17	41.49	100.00	100.00
Word Problems				
Addition	95.83	9.52	100.00	100.00
Subtraction	85.42	34.51	100.00	100.00
Multiply	81.92	36.79	100.00	100.00
Division	67.71	43.29	100.00	100.00
Fractions	84.38	35.97	100.00	100.00

Discussion

The goal of the new curriculum was to get students thinking in numbers, symbols and their relationships or, said another way, thinking in patterns. Did practicing the patterns work? The results suggest that the answer is "yes."

Doing more than Common Core

Taught from kindergarten with *Only the NUMBERS count*©, second graders exceeded Common Core State Standards for Mathematics (2011). According to the standards, multiplication, division and fractions are skills to be acquired in third, and not in second, grade. Similar results were reported for kindergarten and first grade, where students met standards a year beyond Common Core requirements. To understand why the new curriculum worked so well, we consider the contributions of the fundamental components which helped students become highly proficient finders and makers of mathematical patterns.

Contribution of the count: Thinking in base-10

Asian children, who use an explicit base-10 count, have long out performed American children in numerical tasks (OECD, 2012; Siegler & Mu, 2008). Similarly, kindergarteners and first graders taught with *Only the NUMBERS count*© and immersed in an explicit base-10 count outperformed comparison classes in the categories of place value, addition and subtraction. By the end of second grade, the students in the current study had been using the count for three years. They were experts in its iterative patterns and the ways in which numbers combine, recombine and decompose to make other numbers using decades and units.

By itself, the count could not have produced these results. Rather, it had to be embedded in a pedagogy that privileged decomposition (Laski & Yu, 2014). The new curriculum not only decomposed numbers, it also decomposed multiplication, division, and fractions.

Contribution of the charts and blocks: Decomposing and practicing the patterns

In kindergarten and first grade, children used the Count-and-Combine Chart and its moveable blocks to decompose numbers in the explicit base-10 count, and to practice the patterns of base-10 solutions for addition and subtraction. In second grade, the patterns multiplied. The intersection of row and column on the Multi-Operation Chart made the related patterns of multiplication, division and fractions visible and concrete. Children decomposed multiplication problems into division problems and division problems into fraction problems. At post-testing, children performed the operations without charts or blocks. Deliberate practice—focused, daily, incrementally increasing in difficulty—played a major part in turning patterns (numeric and operational) into procedures that the children understood.

Anticipated Questions

Did children also use the standard American count? Yes, they could refer to a number, say 11, as “eleven” and as “ten-one.” This was expected: fluency in a second language, in this case, the count, is directly related to the age at which a child is immersed in that language (Johnson & Swain, 1997). All children in the study learned the explicit base-10 count in kindergarten. For them, being fluent in two counts also meant knowing in which context (addition, time, dates, etc.) one or the other count was appropriate.

Were there any independent assessments of the children’s performance? Yes. All students took state-wide computerized *Renaissance STAR* tests (see Figure 5).

The tests are scored in X.Y format: X is the grade level, Y is the month of the school year. Tests were taken in the eighth month (April) of the school year. A score of 2.8 indicates performance at second grade, eighth month mastery level. Figure 5 shows how second grade scores were distributed across grade levels at that time. Most scores clustered at third grade level, which reflects the mean score, 3.7, indicating expected performance during the seventh month (March) of third grade. Several children scored at fourth and fifth grade levels.

How well did students who were not taught with *Only the NUMBERS count*©? Since this was the only school using the new curriculum, the scores shown in Table 5 provide evidence that students in other schools did not do as well. Grade levels are based on state-wide results. Eighty-three percent of students in the study scored above grade level in second grade.

What about teacher effects? Children were distributed in different classes with different teachers in kindergarten, first and second grades. To assess how the second grade teachers compared with each other, *Renaissance STAR* means for their entire classes at 2.8 (second grade, eighth month) were examined. The scores included students in the study and students who only used the new curriculum for one or two years. Two class means were 3.4, the third was 3.1. These scores are below the 3.7 mean for students in the study, but are still above grade level, indicating that all teachers were effectively using the new materials.

Limitations

The present study lacked both a large sample size and a traditional matched comparison group (i.e., a specific class or school using a different 2nd grade curriculum). The reasons were: (1) only 24 students taught with the new curriculum from kindergarten through second grade were tested, and (2) given the program’s success in kindergarten and first grade the principal of the participating school requested that all second grades use the new curriculum.

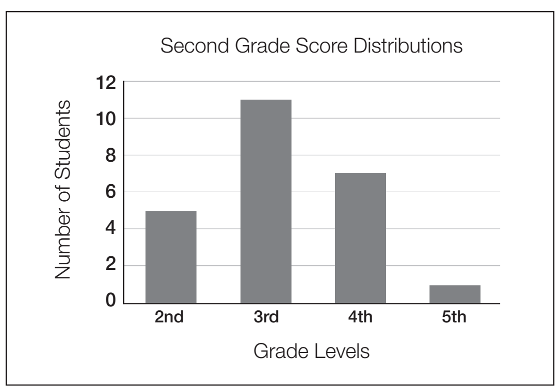

Figure 5. Second grade score distributions on Renaissance STAR tests.

Educational Implications and Conclusion

This study has three important implications for mathematical education and future research.

First, thinking and problem solving almost exclusively in patterns composed of numbers and symbols can help young students acquire both the procedural (how-to) and conceptual (that/what) knowledge required for continued mathematical achievement. Evidence for this includes the following. Earlier work showed that daily practice decomposing an explicit base-10 count, helped students learn how to combine and re-combine a small set of numbers (1 to 10) in regular patterns, and also understand what those combinations meant (place-value). By making "combinations" on the Count-and-Combine Chart with individual sets of "blocks," students learned how to do addition, which reinforced understanding that numbers are combinations of other numbers. Taking away the same number of blocks from both sides of the equals sign, they learned how to do subtraction and to understand that subtraction "un-does" addition. The current study showed that working with the inter-related patterns on the Multi-Purpose Chart allowed students to learn how to do multiplication, division, and fractions. They also understood the relationships between the three concepts, for example: that both multiplication and division involve groups of equal size, that division "un-does" multiplication, and that unit fractions can represent division.

Two, an impactful curriculum can be inexpensive and easy to teach. All the materials were made by the teachers with foam core, VELCRO©, and poster board. All lesson plans were included in two handbooks. Progress depended on what happened in a particular classroom, not on a strict succession of day-by-day work sheets. Importantly, progress depended on how enthusiastically teachers adopted the curriculum. All the teachers said adoption was easy and for two major reasons: the materials were intuitive and accessible; the students were engaged and eager to use them.

Three, regardless of whether the Mathematical Standards set by the Common Core (or their state-by-state replacements) are taken to represent a floor or a ceiling for student achievement, the current results suggest that those standards underestimate the potential of early elementary school students. While the current results clearly require replication with large paired comparison groups from multiple schools in diverse settings, they strongly support arguments (Laski & Yu, 2014; Vasilyeva et al., 2015) that differences in performance between Asian and American students rest on differences in the count and the curriculum in which it is embedded. Like Asian schoolchildren, the second grade students had early and extensive experience decomposing numbers using an explicit base-10 count. Unlike American schoolchildren, they were expected to master basic multiplication, division, and fractions and their relationships a year earlier than required by American curricula.

The conclusion, which would be far stronger with replication, is that the mathematics standards set by the Common Core can, and should, not only be achieved, but exceeded. The students in the current study did more than expected by the Common Core. Because they can, we should expect more.

References

Chase, W.G., & Simon, H.A. (1973). The mind's eye in chess. In W.G. Chase (Ed.), *Visual information processing* (pp. 215–281). New York: Academic Press.

Chi, M.T.H. (2011). Theoretical perspectives, methodological approaches, and trends in the study of expertise. In Y.Li (Ed.), *Expertise in mathematics instruction: An international perspective* (2nd Edition, pp. 17–39). NY: Springer.

Chi, M.T.H., Glaser, R., & Farr, M.J. (1986). *The nature of expertise.* Hillsdale, NJ: Erlbaum.

Common Core State Standards. (2011). *Common core state standards for mathematics.*

Ericsson, K. A. (2006). The influence of experience and deliberate practice on the development of superior expert performance. In K.A. Erricson, N. Charness, P.J. Feltovich, & Hoffman, R.R. (Eds.), *The Cambridge handbook of expertise and expert performance* (pp. 683–704). NY: Cambridge University Press.

Ericsson, K.A., & Kintsch, W. (1995). *Long-term working memory* (Institute of Cognitive Science Tech. Rep. No 94-01). Boulder: University of Colorado.

Fuson, K.C. (1990). Conceptual structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction, and place-value. *Cognition and Instruction, 7,* 343–403.

Fuson, K.C., & Kwon, Y. (1992). Korean children's single-digit addition and subtraction: Numbers structured by ten. *Journal for Research in Mathematics Education, 23,* 148–165.

Johnson, K.J., & Swain, M. (Eds.) (1997). *Immersion education: International perspectives.* NY: Cambridge University Press.

Jordan, N.C., Kaplan, D., Ramineni, C., & Locuniak, M.N. (2009). Early math matters: Number competence and later mathematics outcomes. *Developmental Psychology, 45,* 850–867.

Laski, E.V., & Yu, Q. (2014). Number line estimation and mental addition: Examining the potential roles of language and education. *Journal of Experimental Child Psychology, 117,* 29–44.

Miura, I.T., & Okamoto, Y. (2003). Language supports for understanding and performance. In A.J. Barody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Developing adaptive expertise* (pp. 229–242). Mahwah, NJ: Erlbaum.

Nokes, T.J., Schunn, C.D., & Chi, M.T. H. (2010). Problem solving and human expertise. In *International Encyclopedia of Education,* Vol. 5 (pp. 265–272).

Organization for Economic Co-operation and Development (OECD) (2012). PISA 2012 technical report. <http://www.oecd.org/pisa/>

Siegler, R.S., Duncan, G.J., Davis-Kearn, P.E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M.I., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science, 23,* 691–697

Siegler, R.S., & Jenkins, E. (1989). *How children discover new strategies.* Hillsdale, NJ: Erlbaum.

Siegler, R.S., & Mu, Y. (2008). Chinese children excel on novel mathematics problems even before elementary school. *Psychological Science, 19,* 759–763.

Simon, H.A. (1988). *Learning from examples and by doing.* Invited talk, presented October 3, 1988 at Teacher's College, Columbia University, NY.

Soloway, E., Adelson, B., & Ehrlich, K. 1988). Knowledge and processes in the comprehension of computer programs. In M.T.H. Chi, R. Glaser, & M.J. Farr (Eds.), *The nature of expertise* (pp. 129–152). Hillsdale, NJ: Erlbaum.

Stevenson, H.W., Chen, C., & Lee, S. (1993). Mathematics achievement of Chinese, Japanese, and American children: Ten years later. *Science, 259,* 53–58.

Stokes (2010). Using constraints to develop creativity in the classroom. In R. A. Beghetto & J.C. Kaufman (Eds.), *Nurturing creativity in the classroom* (pp. 88–112). NY: Cambridge University Press.

Stokes (2013). The effect of constraints in the mathematics classroom. *Journal of Mathematics Education at Teachers College, 4,* 25–31.

Stokes (2014a). Using a creativity model to solve the place-value problem in kindergarten. *The International Journal of Creativity & Problem Solving, 24,* 101–122.

Stokes (2014b). How early is early enough? Solving the place-value problem in first grade. *The New Jersey Mathematics Teacher, 72,* 30–40.

Vasilyeva, M., Laski, E.V., Ermakova, A., Lai, W-F., Jeong, Y., & Hachigian, A. (2015). Reexamining the language difference account of cross-national representations in base-10 number representations. *Journal of Experimental Child Psychology, 129,* 12–25.

Zimmerman, B. J. (2006). Development and adaptation of expertise: The role of self-regulatory processes and beliefs. In K.A. Erricson, N. Charness, P.J. Feltovich, & Hoffman, R.R. (Eds.), *The Cambridge handbook of expertise and expert performance* (pp. 705–722). NY: Cambridge University Press.