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Forward-Thinking Orientations for Mathematics Education

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Building Thinking Classrooms: A Conversation with Dr. Peter Liljedahl

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ABSTRACT This conversation is a slightly modified version of an interview with Dr. Peter Liljedahl, Professor of Mathematics Education. The interview was conducted by *JMETC* Guest Editors on March 8, 2021, and was based on Dr. Liljedahl's December 7, 2020 colloquium presentation "Building Thinking Classrooms: A Case of Results First Research" at Teachers College, Columbia University.

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Introduction

Dr. Peter Liljedahl is a Professor of Mathematics Education at Simon Fraser University in Canada. In this dialogue, he talks about his experience and research on classroom practices that provide opportunities for students to think deeply about mathematics. His new book *Building Thinking Classrooms in Mathematics, Grades K-12: 14 Teaching Practices for Enhancing Learning,* is a practical guide for teachers based on 15 years of research. Dr. Liljedahl discusses evidence-based practices from the book. He offers suggestions for how teachers can adapt his framework to teaching online during the pandemic and comments on building equitable thinking classrooms.

JMETC: Please get us started by sharing your favorite mathematics teaching memory.

Dr. Liljedahl: This is actually a memory that is critical in my growth and realization that what's happening in schools is not really that beneficial to students, sometimes. I was teaching a calculus course at a high school. I felt it was important to create and teach a calculus course for students who were most likely to be shocked by their experiences in calculus the subsequent year when they got to university. So, not the top-top students in the school—they were taking AP Calculus. This was the next tier of students. I had been teaching the students for several months, and I was moving through the curriculum and teaching in what I would say was a relatively traditional way.

We were about to start doing implicit differentiation of exponential functions, and they had all of the tools to be able to arrive at the process themselves. We had done implicit differentiation of trig. functions. They also had familiarity with exponential functions from the concurrent Math 12 course that I was teaching them. I thought I would do something different this morning. I wrote a question on the board, and I said, "Try to figure out what the derivative of this function is. Talk to the people sitting around you and see what kind of progress you could make." And they all sat there and stared at me. And you know, as a teacher, when you have that dead space when you've asked students to do something, and they're not doing anything, those seconds feel like hours. Finally, I said, "I have to go do some photocopying," and I left the room and stood outside to watch what was happening inside. Nothing was happening. They weren't talking. They weren't trying. They just sat there.

After about ten minutes of me standing out in the parking lot watching, I came back, and I said, "Okay, does anyone have any thoughts, is there any progress made on this task?" They just sat there and stared at me. After maybe a minute of this waiting, I said, "I have to do some more photocopying," and I left. And I stood out

of the parking lot for another ten minutes and watched. And nothing was happening. So, I came back inside and asked, "Anything?" And they just stared at me, so I said, "Class dismissed." And their jaws just dropped. They were kind of in shock.

They came in two mornings later and the exact same question was written on the board at the front of the room. As they settled, I just said, "We're still on the same question. Talk to the people sitting around you, see if you can figure anything out, make any progress; I have to go do some photocopying." I left the room and stood out in the parking lot. I stood out there for 25 minutes before I actually saw them start talking and discussing. Then, after about ten minutes, I came back in, and I said, "Anything?" Now they had things to offer. For me, this was an awakening that we're doing students a disservice if we just chop curricula into small bite-sized pieces and feed it to them. Although I had given my students all the tools to be able to answer this question, they didn't know how to do anything on their own in a novel situation. They were waiting for me to spoon-feed them.

A few days after that, I announced that the upcoming unit test was going to be a take-home exam. They were to show up on Friday after school and would have the whole weekend to work on it. The last question was, "How many hours did you spend working on this exam?" When they came into class on Monday, they were in a really bad mood. But it was an interesting bad mood because they all sort of said the same thing: "That was way too hard. But we think we know what you mean."

My argument was, just because you don't know how to do something immediately, it doesn't mean you don't have the tools to do it. The actual purpose of this exam was to push them into that space. I asked questions that they had never seen before, all of which they absolutely had the tools to answer. They were allowed to use their notes. They were allowed to use a textbook. They had all the resources. But they were still going to have to apply these in ways that were going to require them to think. The average amount of time that they had spent on that exam was 20 hours. In the end, I gave them a choice of whether they wanted me to count that exam or not. The point was, I was trying to break them of this habit of just waiting for me to tell them how to do things.

I know what calculus looks like at university. It's three times as fast, and they cover more content, and there is no sympathy or empathy for a student who isn't keeping up. The goal of this experiment—my endeavor in this moment—was to get them to realize that moving forward in math is not about being shown 100 recipes on how to do 100 different things. It's about being shown a small set of fundamental principles and then figuring out how to apply them in novel situations. That was an eye-opening experience for me. It was, interestingly, not this experience that contributed to me starting this work on building thinking classrooms. But, as I was doing that work on building thinking classrooms, I did reflect back on that experience many times and realize that in some sort of a parallel-universe way, I was now working on the same problem I had encountered all those years previously.

JMETC: Where did your idea to wait and give them time to think over two days come from?

Dr. Liljedahl: I'd like to say that it was something deterministic, that I came into the lesson that day determined to do this. No; it emerged in the moment. I put up a question that I believe that they should be able to do, and I was met with silence. I don't know why I decided at that moment that I was not going to cave–that, rather than stand there and have that uncomfortable silence between us, I was going to leave the room. That was just to break that tension—that tension of them waiting for me and me waiting for them.

When I think back to my years as a student in university learning mathematics, I remember my third-year advanced algebra course (rings and fields): the instructor would give us homework assignments, and we had two weeks to submit them. Each homework assignment was usually six questions. It was not a large number of questions, and we had two weeks, but I remember that I would come home, and I would start to work on it, and after four hours of work, I would have no answers. But, I tried every question a little bit. I had made no progress towards an answer on any one of the questions, but when the two weeks were up, I always was able to submit a complete homework assignment. I realized in that time that what separated me from having an incomplete homework assignment to having a complete homework assignment was always just effort. It was effort and thinking, and persevering, and trying to find ways forward with the tools that I had.

I realized that this is what these students were missing. If they were going to university the next year and most of them were going to take Calculus I and II in their first year, this was what they needed. So, there were a lot of ideas happening at the same time. But I think, to be honest, at the moment, it was just organic.

JMETC: Would you please share some important highlights from your work?

Dr. Liljedahl: Building thinking classrooms is something that I've been working on for the last 18 years. It culminated in the publication of a book called *Building Thinking Classrooms in Math.* I think the clearest summary I can give is that this project is a reaction to a realization that in many classrooms I visit, students are not thinking. When I spent time at the beginning of the project in numerous classrooms, I gathered baseline data around how much time students spend thinking within one lesson. And, when I say thinking, I mean thinking in ways that we know students need to do in order to be successful in future math courses. What I'm not talking about is mimicking—that sort of regurgitation of what they've just been shown. I think there are many people out there who have had the experience of being mimickers themselves and then at some point hitting the wall in high school when the mimicking stopped working—when all of a sudden the questions became too intricate for a student to just be able to regurgitate something.

So, I was looking for thinking behavior as opposed to mimicking behavior. The baseline data showed that in a typical lesson—and this was true from kindergarten up, although primary always behaves a little bit differently—approximately 20% of students spend approximately 20% of the lesson thinking. The other 80% of the students spend zero time thinking. There's a small percentage of students who do a little bit of thinking when afforded the opportunity, but the rest of the students spend all of their time not thinking. They're mimicking, stalling, slacking, faking, but they're not thinking. Building thinking classrooms is a reaction to the non-thinking reality that I was seeing. This non-thinking behavior explains a lot of why we have troubles in mathematics around the world; it's not a uniquely Canadian issue or uniquely American issue. This may be the explanation: If students aren't thinking, they're not learning. Period! So, how could I get more students thinking, and for longer than they were in this baseline data?

That led me down a 15-year project of doing research into teaching practices. How can we do things differently that will afford and enable and require students to do more thinking within our lesson? Originally this work was chaotic, but was eventually organized around the core practices that every teacher does. There are 14 core practices that all teachers enact, and this emerged empirically from the spending time in classrooms. Every teacher uses tasks of some sort. Every teacher uses collaborative groups in some way. Every teacher has students do their work somewhere—the students have to do the work somewhere. We assign homework, we give notes, we do formative and summative assessments. We answer questions, we give hints and extensions, we consolidate lessons. And those became the variables around which we explored and organized this research. Are there practices within these 14 variables that will generate more thinking? It turns out there are.

In hindsight, it's not hard to understand why traditional teaching doesn't produce this kind of thinking. If we look back at the origins of public education, it was forged at the end of the first Industrial Revolution. It was built on some core principles, but it was really designed to create conformity and compliance. The practices that were established one hundred and fifty years ago have persisted in classrooms today. But our goals are different now; our goals are not to create conformity and compliance. We have these 21st-century learning goals; we're supposed to make students be critical thinkers and creative thinkers who are thinking outside of the box. We're supposed to address issues of equity and diversity. And yet, we're holding on to practices that were born in the crucible of conformity and compliance. How can we possibly expect those same practices to achieve these new goals? When we disregarded this convention and tradition and history and just looked for empirical evidence of how much thinking was being done, and how many students were thinking, what emerged were very different practices.

In a traditional normative classroom, students sit, the teacher stands. The teacher writes on the board, students write in a notebook. The teacher demonstrates how to do a task. The students replicate how to do that task. This is very conventional teaching and has been with us for 150 years. In a thinking classroom, students stand and write on the boards. They work in groups of three that are randomly selected. They work on tasks that they have not yet been shown how to do. Visually, this is radically different from how normative classrooms work. But there's more to it: There's lots of nuance in everything from how we give a task, to when we give a task, to how we arrange the furniture in our classroom, to how we answer questions, to how we assign homework, to how we consolidate the lesson. Every one of these things is very different. One of the things that emerged out of this research was that everything matters. How we do every little thing in a classroom makes a difference to whether we promote thinking or enable non-thinking behavior.

JMETC: When we publish this issue, teachers will still be teaching during the pandemic. What advice do you have for building a thinking virtual classroom, particularly when it comes to being intentionally less helpful—one of the optimal practices that you recommend for building thinking classrooms?

Dr. Liljedahl: We've had a year to play with this. I can't believe it's almost exactly a year now. Just as my book was ready to be released, which is 15 years of research in face-to-face classrooms on how to teach in face-to-face classrooms, we go online. So, we spend time thinking about and exploring and experimenting with which of these practices are still transferable from face-to-face to the online environment and which ones are either not at all applicable or need to be adapted. Surprisingly, a lot of things transfer with minor tweaks or with no tweaks at all. And I think the reason for this is that, fundamentally, we're still trying to achieve the same goal. If thinking is a necessary precursor to learning, whether we're online or face-to-face, we have to find ways to get students to think. If we want students to think, number one, we have to give them something to think about. We have to give them a thinking task. What defines a thinking task? Thinking is what we do when we don't know what to do. So, we still have to give a task where they don't know what to do. Which means we can't pre-teach how to do it. We give them something to think about, we give them someone to think with—so we're still going to put them in groups to do that thinking. We need to give them somewhere to do the thinking. So, we need to give them a space where they can share notation and share representation.

This is true in a face-to-face classroom and it's also true in an online setting. So, how are they different? In a face-to-face classroom, we have such ubiquitous access to resources that I think we've always taken for granted. Whether students can sit on the floor, or they can go and get manipulatives, or they can write on a small whiteboard or they can use technology, there's that ubiquitous access to resources when doing a task. All of a sudden, when we went online, we lost a lot of resources, and that ability to use resources to do a task. So, one of the things that we have to do is to make sure that the task we assign matches in some way the collaborative space that students can work in. If our online environment allows students to talk to each other through Zoom or through some other visual collaborative space, like you and I have right now, then that sets up a kind of situation that allows for certain tasks. If, in addition to this, they have a representational space where they can draw and show their notation, that opens up more possibilities for tasks. If the only thing you and I have is some sort of online asynchronous forum, that determines a different type of task that we can use. So, the digital space in which the teaching and learning is happening, in many ways, has to be taken into consideration when we're picking the thinking tasks. I think a task I would use in a face-to-face classroom may not work in an online classroom, so I have to be careful about how I pick my thinking task.

In a face-to-face classroom, we learned that the optimal group size is three—two if it's in a primary lesson. This has to do with the idea that in order for a group to be generative, there has to be a balance of both diversity and redundancy—this comes from complexity theory. Redundancy is the common knowledge or the prerequisite knowledge that we share. That knowledge is necessary for you and I to even begin to communicate. If you're talking about race cars in French and I like talking about fishing in German, we're not going to have a lot to talk about. We need to have something in common in order to just kick start the conversation. But if all we have is what's in common, then there's no advantage to working in a group. We need to also have diversity, which means that I have to know and think about things that you don't, and vice versa. It is this diversity that allows a group to be greater than the individual. Why are groups of three so optimal? Because the bigger the group gets, the harder it is to have redundancy. But the bigger the group gets, the more diversity you get. The smaller the group, the more likely you are to have redundancy, but you're going to have less diversity. So, groups of three seem to have this perfect balance between redundancy and diversity. The problem is that when we put three students online in a collaborative space, one of them has technology issues, the other one does not want to turn on the microphone or their camera. All of a sudden, the diversity just gets depleted. We found that when we went online, we needed to increase the size of the group to five. We needed to artificially keep that diversity up because online spaces are diversity-depleting spaces. When the group starts to really sync, and students become more open to having their microphones and their cameras on, and they solve all their technology issues, we can start to go back down to groups of three. But, initially, we need larger groups.

Then, we need a collaborative space; we need a space where we can actually share notation. Talking and waving our hands around a lot is good for some tasks, but once we get into more challenging mathematics, we need to actually be able to draw graphs, solve equations, and share notation. So, how do we find those spaces?

Well, things like Zoom have a collaborative whiteboard that we can use. There's a number of third-party digital whiteboards that can be used for groups to collaborate, and my favorite is Jamboard. The nice thing about Jamboard is you can use it with any collaborative space, whether we're in Zoom or Google or Teams. Even if we're on a phone or Facetime or Skype—haven't heard that term for a long time—we can open up a shared Jamboard and work on that. So, those are some of the big things that needed to be changed.

Now, you asked, in particular about being deliberately less helpful. In a face-to-face setting, being deliberately less helpful is actually a move that necessitates that the group start to collaborate more with each other—it's a way to increase knowledge mobility. In a traditional setting, knowledge transmission always happens from teacher to student. If we have 30 students sitting there, all knowledge transmission is happening in one-way lines from the teacher to each student. In a thinking classroom, we have these groups of three working together, so knowledge is transferring between those three students. But they're standing next to another group of three and we need knowledge to move between them as well. We need knowledge to move between this group and the group that's on the other side of the room. This is possible because they're writing on a vertical whiteboard. You can see what they're doing. We need knowledge mobility to happen. But knowledge mobility doesn't happen if the teacher holds on to this idea of always being the source of knowledge. If every time a group puts up their hand, and you go in, and you answer their question or give them the next question, they don't need to tap into the knowledge that's in the room. So, this idea of being deliberately less helpful is a way to force them to capitalize on the knowledge that exists around them. And there's other nuances to that practice.

What's the equivalent of that when we go online? Well, it turns out if we're using Jamboard, for example, the students can actually flip between the frames on Jamboard to see what other groups are doing. By being deliberately less helpful, I can force them to do that, but there is another element to this. In a face-to-face, thinking classroom, a group of students who gets stuck on a task will very quickly just look over their shoulder to see what a group is doing next to them, and maybe they'll see "Oh, they've made a graph" or "I like the way they organize their table" or "Huh, that's interesting, what they're doing there," and if it's interesting enough, and they don't understand it clearly, they may turn and talk to the other group.

What's the online equivalent of being able to just look over your shoulder? It's hard to do. I can't jump from one Zoom room to another as a student. I can go look at the Jamboard, but still, I can't communicate with them; I can't hear what they're saying. So, to help with this, we created another type of document. It's called a knowledge feed, and it's usually just a Google Doc that all groups can access. On this Google Doc, both the teacher and students can paste screenshots , so a group might say, "We found an interesting representation" or "We think we have a nice graph," and drop it in. Or the teacher might take a screenshot to capture what a group is doing and say, "Look at the way they organize their data." This knowledge feed is very much like a Twitter feed—it's this constant stream of ideas and knowledge. This needs to be there in order for the teacher to even begin to be deliberately less helpful. So, when a group asks a question, the teacher can just say, "Did you check the knowledge feed?" Rather than wait for the teacher all the time, if a group is stuck or ready to move on, they can go to the knowledge feed to get an idea, to get some inspiration, or get the next question.

JMETC: Could you talk about how your ideas for building thinking classrooms also help to build equitable classrooms? Do these ideas work for all classrooms and for all students?

Dr. Liljedahl: Recently, people have been talking about thinking classrooms through the lens of equity. I've received a lot of accolades for my attention to equity, and of course, I believe equity is important. But, equity was not one of my research questions when I was pursuing and building thinking classrooms. My research question was, can I get more students to think, and can I get them to think for longer? I think equity has become a byproduct of this process. This doesn't sound very nice because I should have been paying attention to that deliberately, but that wasn't my goal at the time.

So, the question is, where are the opportunities for equity to be created within the thinking classroom? And there are a number of them. I think that the most critical example is how students are grouped. Students are grouped randomly. And not just randomly, but visibly randomly, so that the students can see that there's randomness at play—for example, they might come up and pull a card, and all the fours are going to be together. When we interview students about this, they say that if the teacher is willing to throw this to randomness, then it means we are all the same. What we're communicating to students when we assign groups randomly is that nobody in this room is so special that they have to be paired with some other person, either in order to sup-

port them or to be supported by them—we are confident enough in all your abilities that it doesn't matter who you work with. We're all the same. I think this is one of the things that people are attending to when they start to say things like "Thinking classrooms helps with equity." I'm not claiming that because I haven't done the research on it. But, I think this is what they're attending to when they're making these sorts of comments.

There are other things, and again, I'm only talking to you through the lens of what I hear from students. One of the things around the practice of how we answer questions is that students ask three types of questions. They ask *proximity* questions which are the questions they ask just because the teacher happens to be close. They ask *stop-thinking* questions—"Is this right?" and "Are we doing this right?" And they ask *keep-thinking* questions which are extension questions or clarification questions to enable them to get back to work. We spent a lot of time trying to pay attention to the types of questions students asked and then gave an appropriate response based on the type of question. What we found was that the absolute best response when a student asked a proximity question or stop thinking question was to smile, nod, go "Interesting," and walk away.

When we interview students in the first few days of experiencing this kind of response, they are generally irritated—"The teacher just walked away, they're not answering our question." But when we interview students three weeks into this experience an interesting phenomenon occurs. We ask the students, "What just happened? You asked a question and the teacher just walked away what does that mean?" The students say, "Well, that means she thinks we can do it." And then we ask, "Well, do you think you can do it?" The students say, "Yeah, we probably could; she's been right before." So, it's again this idea that without any bias, the students are picking up on this idea that the teacher believes that they can do it.

The ninth practice is how we give hints and extensions. For this, the research was heavily informed by Mihaly Csikszentmihalyi's theory of flow and the idea that we have to keep students in a balance between their ability and the challenge of the task. We keep them in balance in how we give hints and extensions. As a group's ability increases, we extend the task, we increase the challenge of the task. And if the challenge is too great for their ability, we give them hints in order to maintain that balance. This is actually a form of differentiated instruction because the teacher is catering to the explicit and immediate needs of a group, whether or not that need is to increase the challenge or increase the ability. They're working with the group to see where they are in that moment, then deciding whether or not that group needs an extended task or if it needs some help.

But this is a very different form of differentiated instruction from what we typically see as differentiation. In this form, all groups start on the same task, and what's differentiated is the pace through which they move through a sequence of tasks and the degree to which they are supported in that venture. The differentiation happens in real time, based on real data of how that group is functioning in the moment. Classically, differentiation is based on an assumption of how an individual or how a group will perform that day and, based on that assumption, the student or the group will be given a different task. So, both forms of differentiation are about meeting students where they're at, but classic differentiation is predicated on assumption and bias and presupposition. What do we think this group will do here? What do we think this individual is capable of? And one of the things that we kept seeing in thinking classroom research was, what a teacher *thinks* a group is capable of is rarely reflective of what they're actually capable of. If we put three students together, often the teacher will say, "That group's going to struggle," and then ten minutes into it, this group is leading the class—they're tearing through the material. So, I think again, this is an equity move in the sense that, rather than assume that we know how a student is going to perform, let's actually see how they perform—and then differentiate our support around that.

But, again, I give the caveat that I'm no specialist in equity. I'm just echoing what I think other people who are experts on equity are seeing in the thinking classroom.

JMETC: Even though it's not something that you set out to do initially, it's important that you've given it some thought. Thank you for sharing.

Dr. Liljedahl: There's a new podcast that was just launched called *The Sum Of It All* (https://anchor.fm/sdcoe-math). It's two people in San Diego, and they're going through the book chapter by chapter, but one of the questions they ask for every chapter is, what are the implications around equity for this chapter? I really like listening to what they have to say because they are asking these hard questions.

JMETC: What plans do you have for building on this work? What's next?

Dr. Liljedahl: I'm working with the publisher to produce a book on building thinking classrooms in atypical settings in math. Online is one of those atypical settings. I think parts of it are going to be with us for a very long time. Also, the idea of hybrid classrooms, blended classrooms, working in a one-to-one situation, or a teacher working with a single group to support unfinished learning. I'm working on what thinking classrooms can look like in these settings.

For many years now, I've also been exploring what thinking classrooms look like in subjects other than mathematics. I've done a lot of work with English teachers, social studies teachers, history teachers, science teachers. How does the framework change when we move into other subject areas?

There's also a real thirst for thinking tasks to go with thinking classrooms and looking at those both from a non-curricular perspective and a curricular perspective. So, there's always something on the go.

JMETC: I'm going to round up by asking you, is there a question that I should have asked you but did not?

Dr. Liljedahl: I think an important question is, how do teachers receive this research or this book? Because the research can come at them in lots of different ways—it could maybe be part of a workshop, they could read the book, they could have bumped into an article on it. Or maybe they encounter a teacher who has encountered the book, and so now they're interacting with a teacher that is excited about what they're playing with and so on. I think there are elements of the thinking classroom research that teachers immediately agree with. I also think there are elements of the thinking classroom that really challenge us as educators because the results in some ways are very contradictory to what we were trained to do, what we were prepared to do, what we ourselves experienced as learners for so many years. It may challenge our very foundational beliefs about what is mathematics and what is important in mathematics. But I think that this is a healthy part of exploring our own practices—to feel challenged.

The thinking classroom framework—these 14 practices and the sequence that it comes with—is just that—a framework. It emerged out of research. I think this framework is enhanced when teachers add their own personality to it. So, it's not a prescriptive sort of recipe—do this, then do that—it's a collection of ideas and a collection of results, and teachers, of course, have to bring their own identity to those results. I often say at the end of a workshop that everything I've told you is guaranteed to fail, and guaranteed to fail in ways that are painful for everybody involved, unless you want it to work. If a teacher wants it to work, then they will find ways to make it work.

I often get asked what is the absolute best quality that a teacher can bring to thinking classrooms. I always have the same answer—if a teacher is willing to be seen as fallible in front of their students, I think that they have all the qualities they need to become a great teacher. If a teacher is willing to be seen as someone who's fallible, who's willing to try things and have them not work, then they have the courage to pursue their own self-study in improvement. The book is just going to help them on that journey.