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JOURNAL OF MATHEMATICS EDUCATION AT TEACHERS COLLEGE

A Century of Leadership in Mathematics and Its Teaching

Making Connections in Mathematics Education

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AIMS AND SCOPE

The Journal of Mathematics Education at Teachers College (JMETC) is a recreation of an earlier publication by the Program in Mathematics and Education at Teachers College, Columbia University. As a peer-reviewed, semiannual journal, it is intended to provide dissemination opportunities for writers of practice-based or research contributions to the general field of mathematics education. Although each of the past issues of *JMETC* focused on a theme, the journal accepts articles related to any current topic in mathematics education, from which pertinent themes for future issues may be developed.

JMETC readers are educators from pre-kindergarten through 12th grade teachers, principals, superintendents, professors of education, and other leaders in education. Articles appearing in the *JMETC* include research reports, commentaries on practice, historical analyses, and responses to issues and recommendations of professional interest.

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PREFACE

The Fall 2019 issue of the Journal of Mathematics Education at Teachers College features five articles that focus on making connections in the teaching and learning of mathematics in order to build deep conceptual understanding, enhance teacher education and reflect on the evolution of doctoral programs in mathematics education. This edition will provide insight on preparing students for advance mathematics through a deeper exploration of basic ideas, highlighting an alternate pathway for success in mathematics at the community college level, effective faculty mentoring programs, and a commentary on improving the experience of graduates of doctoral programs in mathematics education.

In their article, Deihl and Markinson present a method for introducing high school mathematics students to concepts of cardinality in set theory by connecting them to trigonometry. Readers will explore questions and student responses, both real and imagined, which lead students to think about the idea of infinity via the tangent function. Students are exposed to not only the concept of infinity but also bijections between countable sets, bijections between intervals, and cardinality. This represents a wonderful opportunity for secondary mathematics students and teachers to go beyond the required curriculum into some deep and interesting mathematical ideas.

Gil, Zamudio-Orozco, and King designed a study to investigate the kinds of connections that pre-service teachers made in an elementary education methods course. In particular, the authors observed the pre-service teachers' ability to identify over-arching mathematical ideas and the knowledge connections being used to clarify the concepts taught. Based on their findings, the authors show how pre-service teachers' connections changed throughout the course and whether their instructional interventions assisted in these changes. This study shows both teachers and teacher educators how making the right connections can impact mathematical understanding.

In their commentary, Reys, Reys, and Shih take inventory of doctoral programs in mathematics education and offer a critical review of doctoral preparation in mathematics education. They summarize the current state of doctoral students, graduates, programs, and faculty members, emphasizing key areas of consideration and recommending a blueprint for improving the system. The authors conclude by offering practical suggestions for research in various areas of doctoral preparation so that mathematics education can continue to grow and thrive for years to come. This manuscript invites all members—students, faculty, and staff alike—to reconsider their role as a steward of doctoral programs in mathematics education and its influence on research and practice.

PREFACE (Continued)

George and Milman discuss a quantitative study of community college students enrolled in developmental mathematics courses. They studied the use of a quantitative literacy course as an alternative pathway in lieu of elementary algebra. Non-STEM students who had placed into developmental mathematics were connected to an alternative pathway to completing their mathematics requirement. This quantitative literacy course was designed to be more relevant to the students and to use a collaborative student-led model. Students worked on open-ended problems in the areas of citizenship, personal finance, and medical literacy. Passing rates for both the developmental and subsequent credit-bearing courses increased. The authors conclude by recommending that community colleges consider offering such a quantitative literacy course for their non-STEM developmental students.

Finally, Hodge, Rech, Matthews, Johnson, and Jakopovic describe a mentoring program which connected pre-service secondary mathematics teachers to faculty members at a large Midwestern university. The authors tell the stories of the four pre-service teachers who participated in the first year of the program. Guided by the experiences of these four participants, changes were made to the structure of the program regarding mentor selection, mentor-mentee communication, expectations, and how those expectations were conveyed. The changes have made an impact on the program in terms of consistency and implementation. The authors hope that this piece may help guide the creation and improvement of mentoring programs at other institutions.

Together, these five articles highlight the theme of connections in mathematics education. Whether connecting mathematical concepts, educators, or theory and practice, all these pieces provide different ways in which connections can be used to improve mathematics education in elementary school, doctoral studies, and everything in between.

Sarah Nelson
Dyanne Baptiste

Guest Editors

JOURNAL OF **MATHEMATICS EDUCATION** AT TEACHERS COLLEGE | FALL 2019 | VOLUME 10, ISSUE 2

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Connecting the Tangent Function to Cardinality: A Method for Introducing Set Theory to High School Students

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ABSTRACT High school students often ask questions about the nature of infinity. When contemplating what the "largest number" is, or discussing the speed of light, students bring their own ideas about infinity and asymptotes into the conversation. These are popular ideas, but formal ideas about the nature of mathematical sets, or "set theory," are generally unknown to high school students. The authors propose a method for introducing basic ideas in set theory to high school trigonometry students by connecting prior knowledge of the tangent function and the unit circle to Georg Cantor's ideas about infinity. By doing so, high school teachers have an opportunity to inspire their students with rich mathematics.

KEYWORDS *algebra II, asymptote, cardinality, connecting representations, Georg Cantor, infinity, tangent function*

Introduction

Teachers of Common Core Algebra II courses often struggle to cover the scope of the curriculum due to time constraints. The race to finish the curriculum does little to spark students' interest or engagement in mathematics, although modern teaching philosophies suggest "…a movement towards the student being invited to act like a mathematician instead of passively taking in math" (Hartnett, 2017, para. 6). Rather, "…the interested student should be exposed to mathematics outside the core curriculum, because the standard curriculum is not designed for the top students" (Rusczyk, 2016, para. 2).

Inquisitive mathematics students are often heard discussing infinity, asking questions like, "What is infinity plus infinity?," "Is infinity a number or an idea?," "What is infinity plus one?," and "Can one infinity be bigger than another?" Research has shown that students as young as five or six have a vague notion of infinity as an unlimited process, and mathematics educators have attempted to convey the distinction between two different types of conceptualizations. There is a *potential infinity*, such as continually counting from 1, 2, 3, … etc., which is usually the first encounter of infinity for children, and there is a more nuanced concept of an *actual infinity*, which describes a more concrete mathematical entity. This more advanced viewpoint extends the earlier concept of infinitely counting because it "requires us to conceptualize the potentially infinite process of counting more and more numbers as if it was somehow finished" (Pehkonen, Hannula, Maijala, & Soro, 2006, p. 345). While the Common Core State Standards for Mathematics include extension standards that invite investigation and discovery (e.g., "CCSS.MATH.CONTENT.HSF.TF.B.6: (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed" [National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010]), the lack of time in the classroom more often than not restricts engagement in topics that will not be covered on an end-of-year exam.

Cardinal arithmetic can be used to show that the number of points on the real line is equal to the number of points on any segment of that line. The authors of this manuscript, two experienced high school mathematics teachers, conjectured that this notion is highly counterintuitive and baffling for high school students. The authors interviewed 28 students in a group setting in a Common Core Algebra II class in a Title I, New York City public high school. The untracked class consisted of students with varying achievement levels in mathematics. The students in the class unequivocally demonstrated interest and enthusiasm in discussing the nature of infinity. Students' responses to the question "Can you describe infinity?" are summarized below:

- Infinity is an idea that there is an unlimited amount of numbers going from negative to positive.
- Infinity can be beyond time.
- It's a number that never ends; it's an endless number that never stops going, and never stops growing.
- It's not really a number but more of an idea, because a number is just one singular thing and infinity isn't.
- Infinity is not a number, it's like an idea, because the rules for regular numbers don't apply to it. For example, I once heard someone say that infinity times infinity wouldn't be infinity squared, it would stay infinity.

The students even generated their own questions, such as:

- Is it true what he said, that infinity times infinity is infinity?
- Is infinity minus infinity equal to 0?

The authors saw clear evidence that students were interested in the notion of infinity, and the conversation inspired responses even from students who normally remained quiet during class. While it is exciting for teachers to engage in these discussions with their students, topics in set theory are seemingly so far outside of the curriculum that time does not allow for such activities.

Inspired by their interactions with this Common Core Algebra II class, the authors propose enrichment activities in this manuscript. The authors co-developed these ideas based on their teaching experiences and their desires to inspire students with rich mathematics. This manuscript is not a research study, but rather a report intended to propose ways of exposing high school students to some advanced ideas about set theory and infinity and help them reach surprising conclusions along the way.

Prior Knowledge

Before engaging students in the enrichment activities that will be outlined in this manuscript, it is assumed that students will have learned the following topics:

- Similar triangles;
- Functions (including one-to-one functions) and their features (including asymptotes, end behavior, and intervals of increase/decrease);
- Domain and range;
- Interval notation;
- Definitions of the sine, cosine, and tangent of an angle;
- Unit circle, and the fact that for a point (x, y) on the unit circle, $(x, y) = (\cos\theta, \sin\theta)$;
- Quotient identity: $\frac{(\sin\theta)}{(\cos\theta)} = \tan\theta$;
- Unwrapping the unit circle to graph periodic functions (namely, $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$); and
- Visualization of $\sin\theta$ and $\cos\theta$ on an inscribed right triangle in the unit circle as the lengths of the vertical and horizontal legs, respectively.

Specifically, right before facilitating these enrichment activities, the students should learn how to graph the tangent function on the interval $\left[0, \frac{\pi}{2}\right)$ (see Figure 1).

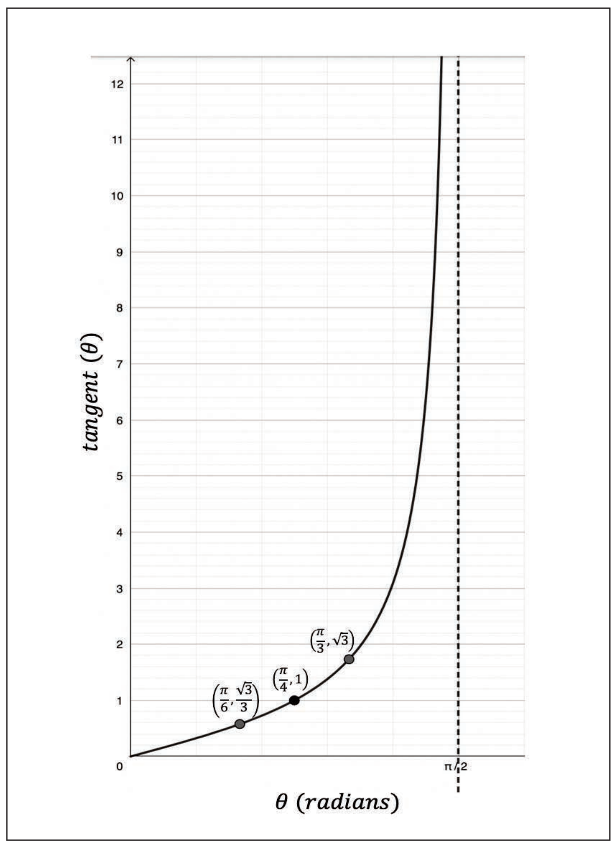

Figure 1. The tangent function on the interval $\left[0, \frac{\pi}{2}\right)$.

Development

To begin our enrichment activities, we will pose the following question to students: Which interval contains more numbers: $\left[0, \frac{\pi}{2}\right)$ or $[0, \infty)$? The students will all think the answer is $[0, \infty)$. However, the following sequence of questions can show students that these sets are in fact equal in size, which they will likely find baffling. Desired responses are indicated in italics.

- Is the tangent function strictly increasing on $\left[0, \frac{\pi}{2}\right)$? *Yes.*
- What is the range of the tangent function on the domain $\left[0, \frac{\pi}{2}\right)$? *$[0, \infty)$.*
- What does the combination of these facts tell us? *The tangent of every angle on $\left[0, \frac{\pi}{2}\right)$ is some real number on $[0, \infty)$. Conversely, every real number on $[0, \infty)$ is the tangent of an angle on $\left[0, \frac{\pi}{2}\right)$.*
- What can we conclude? *Since the tangent function is strictly increasing, every input has exactly one output and every output corresponds to exactly one input. Therefore, the intervals $\left[0, \frac{\pi}{2}\right)$ and $[0, \infty)$ are "equal in size"—they do not have the same length, but they do have equal cardinalities.*

Students may be so overwhelmed by the counterintuitive nature of this idea that they may not "buy it" at first. We propose the following activities to help students understand cardinality and convince themselves that this is, in fact, true.

First, gather the students in the hallway and take them to a classroom they are unfamiliar with. Ask them to devise a way, *without counting,* to determine whether there are the same number of chairs in the room as there are students. Students will surely start sitting down in the chairs—and if every student has one chair, and every chair has one student, they will arrive at the conclusion that there is the same number of chairs as there are students. This leads naturally to the idea that two sets have the same size (or cardinality) if they are in 1–1 correspondence with each other, that is, *if their elements can be matched up.*

Teachers must keep in mind that it is easy to mix up the words "size" and "length" when comparing and contrasting intervals and sets of numbers. It is therefore important to keep reminding students that when we talk about two sets having the same size, we mean that their elements can be paired up until each set is exhausted of its elements. Students will probably have no trouble accepting this, since it is the same way we can tell if two finite sets have the same size (which is exactly what they did when they filled the chairs in the classroom).

Having accepted this, we can next ask if the set of natural numbers is the same size as the set of even numbers. Initially, students will probably all assert that the set of natural numbers is "twice the size" of the set of even numbers. Then, they can be reminded of the definition they agreed upon for "same size" (matching of elements) and can be asked to think about a potential matching between the two sets. Students should come up with the following idea:

natural numbers		even numbers
1	$\longrightarrow$	2
2	$\longrightarrow$	4
3	$\longrightarrow$	6
4	$\longrightarrow$	8
5	$\longrightarrow$	10

In other words, every natural number matches with its double. For students who don't initially grasp the idea that {1,2,3,4...} has the same size as {2,4,6,8,...}, teachers can ask: *If the elements in a set are multiplied by two, does the new set hold a different number of elements, or just bigger elements?* This concrete example can be used to show students for the first time that two infinite sets that appear visually different can have the same size (according to the definition that they agreed upon). This exercise can give them a sense of how powerful the notion of an infinite set can be, and a sense of the mystery of infinite sets.

Immediately following, students can be asked, "Are the set of whole numbers and the set of positive integers the same size?" While initially thinking that whole numbers have "one more element" than the set of positive integers (the element 0), students may be inclined to think of a potential matching between these two sets after having seen the previous examples. Hopefully, students will arrive at the relationship which matches each of the whole numbers with its successor in the set of positive integers. After a few of these examples, students should be willing to believe that different infinite sets can have the same size.

We are about to transition from sizes of discrete sets like the integers, to sizes of intervals. Before doing so, it is important to set up a more mathematically formal definition of matching with the students: Two sets have the same size if there is a bijection between them. The bijection matches x from set A, with $f(x)$ from set B, creating a 1–1 mapping. For instance, the prior two examples can be described by the functions $f(n) = 2 \cdot n$ and $f(n) = n + 1$, respectively. So, the level of the discussion is being raised to include functions. Once students accept the notion that a bijective function between two sets can be used to illustrate a mapping between them, we can ask:

Is a strictly increasing function (or a strictly decreasing function) a bijection between its domain and range? How do you know?

The answer is yes, because if a function is strictly increasing or strictly decreasing it must be one-to-one, meaning that in addition to each input having only one output, each output only corresponds to one input (again relating back to the students and chairs example and the concept of a bijection). We can therefore conclude that for a strictly increasing or strictly decreasing function, the domain and range are the same size. This important result will help students who were hesitant to accept the tangent function example from earlier.

We can bring this example back into focus by again asking:

Does the interval $\left[0, \frac{\pi}{2}\right)$ have the same size as $[0, \infty)$? For students who initially argued "no", we now have a concrete definition that will challenge their thinking and demonstrate that the two seemingly very different sets do, in fact, have the same size.

We can ask the students to think of a graph whose domain on $\left[0, \frac{\pi}{2}\right)$ has a range of $[0, \infty)$. Well, the tangent function is one such graph (as demonstrated in Figure 1). Furthermore, we can use our definition of matching because *the tangent function is strictly increasing on this interval.* Therefore, by the students' accepted definition of "matching," since this function is strictly increasing, its domain and range must be equal in size. Therefore, $\left[0, \frac{\pi}{2}\right)$ is the same size as $[0, \infty)$. Wow!

To take it a step further, you can ask students to compare the sizes of the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the real line, $(-\infty, \infty)$. The students will probably be excited at this point by the previous examples and refrain from incorrectly blurting out that these intervals must have different sizes. They will likely be inspired to come up with their own function to illustrate why these intervals do have the same size—and the astute student might notice that the tangent function does it again! Simply extending the graph in Figure 1 to include negative angles, the tangent function is strictly increasing on the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and therefore matches 1–1 with the range $(-\infty, \infty)$.

This will open students' eyes to the idea that two sets or intervals of finite and infinite extent can actually have the same size. Later in their mathematical studies, students will eventually see examples such as $[0, 1]$ matching with $[0, 1)$ and will then be able to generalize these results and conclude that *any interval has the same size as any other interval, regardless of their lengths and regardless of whether one is open, closed, half open, or infinite.*

In addition to the aforementioned method of convincing students why $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ has the same size as $(-\infty, \infty)$, there is a geometric and visual way to demonstrate why this is true. This can be executed as a discovery learning task in a high school classroom and is detailed below.

Most pre-calculus books show the reason that the tangent function is called the "tangent" function can be understood by doing the following:

Given a coordinate plane with a unit circle and an inscribed right triangle in Quadrant I on it (Figure 2), perform the following sequence of steps. Desired student responses are indicated via italicized font and in the diagrams following the prompts (Figure 3).

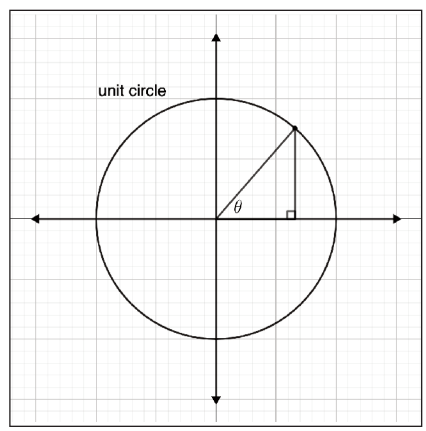

Figure 2. Coordinate plane with a unit circle and inscribed right triangle in Quadrant I.

1. Examine the right triangle that was drawn for you. Label its legs in terms of trigonometric functions of θ, which is the acute angle formed by the radius of the unit circle and the *x-axis.*
2. On the diagram, add the graph of the line $x = 1$. What type of line is this, in relation to the unit circle? *$x = 1$ is tangent to the unit circle. The line intersects the circle at only the point (1, 0).*
3. What kind of angle is formed by the line $x = 1$ and the *x-axis?* Label it on the diagram. *This angle must be a right angle because the segment connecting* (0, 0) *and* (1, 0) *(which is a radius of the unit circle) is horizontal, and* $x = 1$ *is vertical. A tangent to a circle is perpendicular to the radius drawn at the point of tangency.*

4. Extend the hypotenuse of the right triangle so that it meets the line $x = 1$.
5. Locate the vertical segment connecting (1, 0) and the point of intersection of the extended hypotenuse and $x = 1$. Since this vertical segment has an unknown length, label it "a."
6. Separately draw the two right triangles that are now on your picture, copying all information that is known about their side lengths and angles.
7. What do you notice about these triangles? Write a proportion relating the legs of the triangles. Simplify the proportion. *The triangles are similar, since they have two equal angles. The proportion relating their legs is* $\frac{\sin\theta}{\cos\theta} = \frac{a}{1}$. *Simplifying yields* $\tan\theta = a$.
8. What can we conclude? *The tangent of an angle in standard position is equal to the length of the vertical segment connecting the point* (1, 0) *and the point of intersection of the terminal ray with the line* $x = 1$.

After summarizing the conclusion of this activity, present students with the diagram below (Figure 4), which shows the visualization of the tangents of select angles.

Students can be asked, "What happens to the tangent values as the number of radians in the angle, θ, increases to $\frac{\pi}{2}$ or decreases to $-\frac{\pi}{2}$?" As θ increases (or decreases), the points along the line x = 1 will also increase in both directions away from (1, 0) until it becomes impossible to draw them. The tangent values are initially easily located on $x = 1$, but as θ approaches $\frac{\pi}{2}$ or $-\frac{\pi}{2}$, the length of the line tends to ∞ or $-\infty$, as represented by the asymptotes on the graph of the tangent function (Figure 1). This again demonstrates why $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ maps to $(-\infty, \infty)$. Hopefully, students will appreciate the elegant connection between this visual and the sizes of the two intervals in question.

As an extension, students can use Figure 4 to make conjectures about the size of [0, 1] as compared to $[1, \infty)$. They can do this by examining the diagram in the following way:

- Look at the tangent values for angles on $\left[0, \frac{\pi}{4}\right]$. They map to [0, 1].
- Furthermore, the tangent values on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ map to $[1, \infty)$.

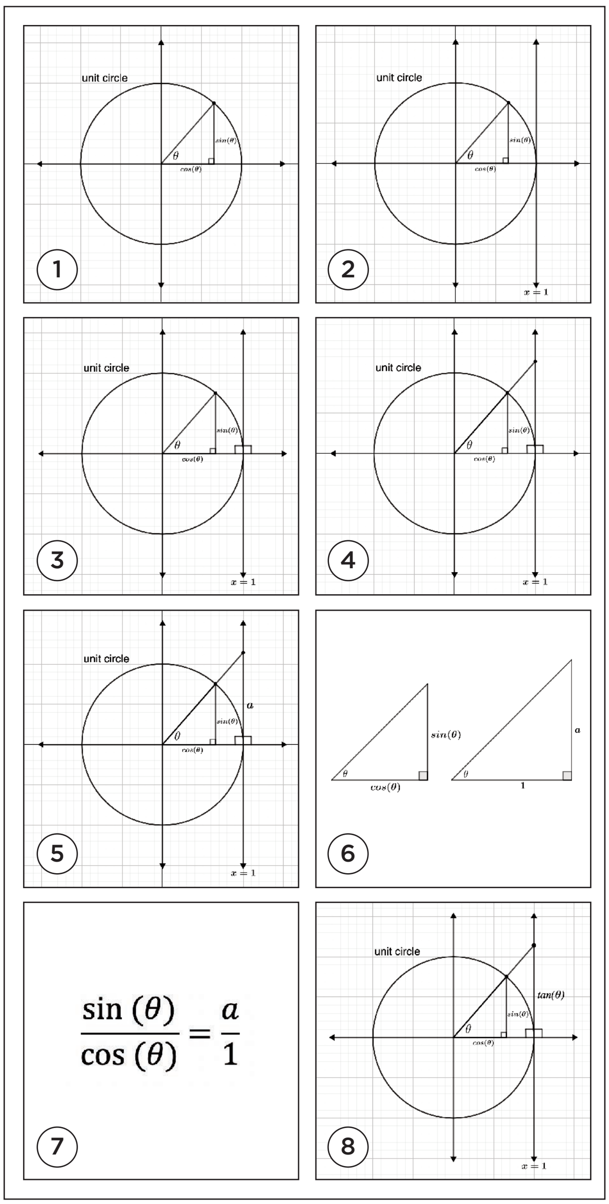

Figure 3. Desired Student Responses.

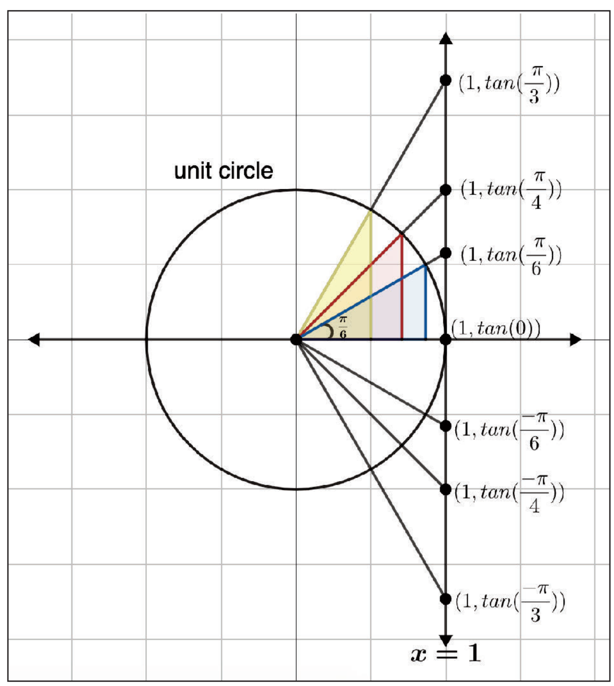

Figure 4. The visualization of the tangents of select angles.

- Since $\left[0, \frac{\pi}{4}\right]$ and $\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ have the same length, and the tangent function is strictly increasing on $[0, \frac{\pi}{2})$, what can be said about the cardinalities of $[0, 1]$ and $[1, \infty)$?

Hopefully, using this diagram will inspire students to seek proof of their conjectures.

Remarks

Our connection to cardinality is made clear through the progression of studying the tangent function and the concept of matching (and, more mathematically, the idea of a bijection). As we encourage students to consider the sizes of two different sets, we are exposing them to the notion of cardinality as well as Cantor's counterintuitive notions of comparing infinities.

As mathematics educators, we recall how amazed we were when first convinced that $[0, 1]$ and $[1, \infty)$ have the same cardinality. Looking back at Figure 4, clearly the sizes of those "segments" on the line $x = 1$ are vastly different. It is expected that students will develop their own questions and ideas for further study, which has great potential to inspire their enthusiasm for learning mathematics.

For teaching purposes, it would be a good idea to provide an example of two infinite sets with different sizes, so that students do not leave this lesson thinking that all infinite sets have the same size.

For additional extension activities, we recommend tasking students with the following:

1. Compare the following sets of numbers. In each case, decide which set is larger or smaller, or whether they are the same size. If you think the sets are the same size, justify your answer by finding a one-to-one function between the sets.
   a. Integers and even integers
   b. Integers and rational numbers
   c. Natural numbers and whole numbers
   d. Rational numbers and irrational numbers
2. What can be said about the number of points on the real line and the number of points on any segment of that line? Justify your answer.
3. Does the tangent function map to every number on the real number line? Explain why or why not.
4. Explain why the cardinalities of $[-1, 0]$ and $[1, \infty)$ are equal using the tangent function.
5. Find a bijection that maps $[0, 1)$ to $[0, \frac{\pi}{2})$.
6. Consider the function $f(x) = \tan(x)$ on the restricted domain $(-\frac{\pi}{2}, \frac{\pi}{2})$. This function illustrates a bijection between its domain and range. How could this function be transformed to illustrate a bijection between $(0, 1)$ and $(-\infty, \infty)$?

For a real challenge, the students can be asked whether the intervals $[0, 1]$ and $[0, 1)$ are the same size. There is a piecewise function that is a bijection between these two intervals, which is an interesting topic for the advanced student to investigate.

Conclusion

In his book *Love and Math* (2014), mathematician Edward Frenkel writes:

> Mathematics is a way to break the barriers of the conventional, an expression of unbounded imagination in the search for truth. Georg Cantor, creator of the theory of infinity, wrote: "The essence of mathematics lies in its freedom." Mathematics teaches us to rigorously analyze reality, study the facts, follow them wherever they lead. (p. 4)

We believe these sentiments should inspire both teachers and students.

The ideas of set theory are accessible to high school students; however, they are almost never taught until college. It is easy to engage students with questions such as, "How many numbers are there between 0 and 1?" and the use of physical models such as the unit circle that they can draw themselves. In fact, previous research has shown that "students use intuitively the same methods [to compare] infinite sets…[and] finite sets. Although students have no special tendency to use 'correct' Cantorian…'one-to-one correspondence,' they are prone to visual cues that highlight the correspondence" (Pehkonen et. al., 2006, p. 346). What is difficult in teaching Common Core Algebra II is finding appropriate places to supplement the curriculum in order to provide enrichment for students and pose interesting and inspirational mathematical questions. A viable option to alleviate this dilemma is to use the tangent function, already in the curriculum, as a launching point to demonstrate ideas about cardinality. This method is visually accessible, rigorous, and innovative. More specifically, it gives high school students seeking enrichment the opportunity to delve into set theory by providing an analogy between the tangent function and notions of infinity.

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References

Frenkel, E. (2014). *Love and Math: The Heart of Hidden Reality.* New York: Basic Books.

Hartnett, K. (2016). Meet the New Math, Unlike the Old Math. *Wired.* Retrieved from www.wired.com/2016/10/meet-new-math-unlike-old-math/

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common Core State Standards for Mathematics. Retrieved from http://www.corestandards.org/Math/Content/HSF/TF/

Pehkonen, E., Hannula, M.S., Maijala, H., & Soro, R. (2006). Infinity of numbers: How students understand it. *Psychology of Mathematics Education, 30,* 345–352.

Rusczyk, R. (2016). The Calculus Trap. Retrieved from https://artofproblemsolving.com/articles/calculus-trap

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After Presenting Multiple Solution Strategies, What's Next? Examining the Mathematical Connections Made by Preservice Teachers

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ABSTRACT When teaching through problem solving, effective mathematics teachers need to lead discussions that assist students in making connections between different solution strategies. However, while teaching a methods course for preservice teachers (PSTs), we noticed that after solving a problem and presenting various solution strategies, many PSTs seemed lost on how to proceed with the mathematics lesson. To address this issue, we designed an action research study where we implemented Smith and Stein's (2011) five practices for orchestrating productive classroom discussions, and focused our attention on the fifth practice, making connections. Specifically, we designed an instructional intervention to examine the type of connections made by PSTs and how these connections changed as the course progressed to aid PSTs' connection making skills. We identified three types of connections made by PSTs: superficial knowledge connections, procedural knowledge connections, and conceptual knowledge connections. Additionally, we observed a decrease in the amount of superficial knowledge connections and an increase in the amounts of procedural knowledge connections and conceptual knowledge connections made by PSTs throughout the course.

KEYWORDS *connections, conceptual, mathematics, mathematics education, preservice teachers, procedural; teacher education*

Introduction

Making connections is central to learning in mathematics (Association of Mathematics Teacher Educators [AMTE], 2017; Bingolbali & Cusknn, 2016; National Council of Teachers of Mathematics [NCTM], 2000). According to the AMTE (2017) standards, it is the teachers' responsibility to lead effective discussions that draw out important mathematical connections from students; however, past research has shown that leading such discussions is particularly difficult for pre-service teachers (PSTs) (Ball, 1993; Lampert, 2001; Stein, Engle, Smith, & Hughes, 2008). Novice teachers do not have a reservoir of mathematical knowledge that can be used to identify connections, particularly at the spur of the moment (Schoenfeld, 1998). Furthermore, many PSTs learned mathematics in classrooms where connections were not emphasized; therefore, they may struggle to make connections and may not understand the important role of facilitating discussions that draw out connections to support student learning. Thus, aiding PSTs in making connections and helping them understand the types of connections that can be used to promote mathematical learning through their teaching must be a vital piece of undergraduate mathematics education courses.

Action Research

After teaching an undergraduate elementary mathematics content and methods course, we noticed a disconnect between our expectations about how PSTs should use connections and what we saw PSTs demonstrate during classroom discussions, in their reflective writing, and while teaching practice lessons in front of their classmates. To address this disconnect, we designed an action research project. Action research is a reflective process led by teachers in their classrooms (Patthey & Thomas-Spiegel, 2013). In this practitioner-centered approach, the teacher designs an instructional intervention for the classroom and evaluates its impact on students (Somekh & Zichner, 2009). In the current project, we created a new curriculum designed to develop PSTs' ability to use connections effectively in the elementary mathematics classroom. Our main goals were to investigate the types of connections PSTs made throughout the course and to evaluate whether PSTs made more effective connections as the course progressed.

Action research serves as a systematic way to improve the practice of educators, especially through its personal and reflective nature, by providing teachers with a method to improve the critical areas they choose to work on (Somekh & Zichner, 2009; Patthey & Thomas-Spiegel, 2013). Not only did we choose action research to help PSTs develop ways to make mathematical connections in problem solving, but also to become better educators. We wanted to learn from the collected data and use this information to further improve our teaching practices. This speaks to the continuous cycle of action research, where constant reflecting, monitoring and modifying are necessary for improvement (Patthey & Thomas-Spiegel, 2013). To engage in this reflective process, we conducted a making-connections activity in class, collected student work, read and analyzed responses, drew conclusions about the connections made, and used this information to construct a new activity for the next class.

We acted as instructors, observers, and participants in this study. We were the instructors, designing and grading all the assignments as the semester progressed. We acted as observers as we took field notes while PSTs presented mathematics problems and role-played different classroom scenarios. Finally, we acted as participants when we provided feedback and facilitated discussions. Because action research is also used as professional development (Oolbekkink-Marchand, Van der Steen & Nijveldt, 2014), playing the roles of observer and participant allowed us to examine the classroom through new lenses. This assisted our overall goal of becoming better educators through action research.

Problem-based Instruction

When using problem-based instruction, students are given genuine problems to solve. That is, problems for which students have not been told specific solution strategies to use and do not perceive there is a single, correct way to solve the problems (Hiebert, et al., 1997). Students create strategies for solving a given problem by building from their prior knowledge. This often requires several attempts at various ways to solve the problem before coming up with a logical approach. After developing strategies, students present and discuss the various strategies that were created. By presenting their strategies, students develop a sense of ownership as they are encouraged to agree or disagree with their classmates to further their understanding of the topic (Chapin, O'Connor, & Anderson, 2009).

Pre-service teachers need a structure to follow in order to create an effective problem-based classroom. One pedagogical method teachers can implement in the classroom is Smith and Stein's (2011) *Five Practices for Orchestrating Productive Mathematics Discussions.* Smith and Stein's (2011) five practices are a set of skills designed to assist teachers who use problem-based instruction. They are described by Smith and Stein (2011) as:

1. Anticipating potential student responses to challenging mathematical tasks;
2. Monitoring students' as they create solutions to the tasks;
3. Selecting particular students to present their strategies for solving the problem;
4. Sequencing the student responses to be displayed in a specific order; and
5. Making connections between the different solution strategies presented to highlight key mathematical ideas.

The teachers' role in the fifth practice, making connections, is critical. To enhance learning, the teacher must identify which mathematical connections they would like to focus on, understand how these connections are related to their learning goals for the lesson, and use students' mathematical thinking to help make these connections explicit (Stein et al., 2008). In our experience of teaching mathematics methods courses, PSTs struggle during this critical phase of a problem-based lesson. To address this, we designed an intervention focused primarily on the last practice, making connections. Smith and Stein (2011) argued that making connections must be at the forefront of teachers' thinking throughout the five stages. For example, the decision about who to select to present solution strategies should be guided by the connections that the teacher would like to emphasize during the whole class discussion. Moreover, with practice, PSTs can steadily improve at facilitating whole classroom discussions focused on making connections, a necessity when unpacking cognitively challenging math problems (Stein et al., 2008). This is where students have an opportunity to further their thinking and develop deep mathematical understanding of the material.

Connections

A mathematical connection is defined by Mhlolo, Venkat and Schäfer (2012) as "a process of making or recognizing links between mathematical ideas" (p. 2). The brain is equipped with the ability to make connections (Caine & Caine, 1991), and this ability can be enhanced when teachers conduct lessons that emphasize, draw out, and formulize the connections inherent in the mathematics being studied. Tchoshanov (2011) found that student achievement increased significantly as teachers' knowledge of concepts and mathematical connections increased. Thus, it is important for teachers to have a deep understanding about mathematics concepts and the relationships that exist between concepts to engage in discourse and questioning that makes connections apparent to students.

Mathematical understanding has been described in terms of instrumental understanding and relational understanding (Skemp, 1976). Instrumental understanding is defined as the ability to carry out procedures and rules without knowing why they work. This understanding is related to procedural knowledge as it requires the use of formal language, symbolic mathematical notations, and rules for completing a mathematical task (Hiebert & Lefevre, 1986). Teachers and students have relied on their ability to use procedures and rules to demonstrate their mathematical understanding. However, instrumental understanding falls short of developing the skill of making connections as the procedures used for specific problems are not always generalizable to other problems (Rittle-Johnson, Siegler, & Alibali, 2001).

Skemp (1976) described relational understanding as understanding which engages students in knowledge about mathematical operations and the reasons for why they work. Further, relational understanding is associated with conceptual knowledge as it consists of an integrated network of mathematical concepts, where each piece of information is connected with other information to develop mathematical understanding (Bingolbali & Cusknn, 2016; Hiebert and Lefevre, 1986; Skemp, 1976). This type of understanding is highlighted in the process of learning mathematics and is constructed in classrooms that emphasize the skill of making connections between concepts and among different representations of a concept.

When teaching to build relational understanding, identifying mathematical connections between solution strategies becomes a central component. For example, as students engage in double-digit addition, teachers can highlight connections by using place value understanding (CCSSO, 2010). As shown in Figure 1, when solving the problem 25 plus 38, a student may use the split method, separating each number by place value.

$$\begin{array}{rr} & 20 + 5 \\ + & 30 + 8 \\ \hline & 50 + 13 \end{array}$$

Figure 1. Split method for solving double-digit addition.

The teacher can use this as an opportunity to demonstrate how the sum of five and eight, both in the ones place, equals 13, a number with one ten and three ones. Further, through discussion and questioning, PSTs can help students connect the split method solution strategy to the memorized procedure of carrying over the ones to the tens place. This leads to meaningful learning because students are able conceptualize the underlying reasons for why procedures work and identify relationships that exist between multiple representations.

Teachers need to have an understanding about the types of connections that exist and how these connections promote mathematical learning. Smith and Stein (2011) argued that making connections is the most challenging practice, as it calls for teachers to identify important connections and create questions that make those relationships explicit for students. Questions that elicit powerful connections build on students' solution strategies by drawing out relationships between various strategies or relationships that exist between strategies and big mathematical ideas. By asking targeted questions, teachers can push students to think deeply about important mathematics.

Although an essential role of the teacher in the problem-based classroom should involve making connections, we know very little about the types of connections PSTs make, and whether their ability to make connections can improve over time with targeted instruction. Therefore, we designed a study to investigate the following questions:

1. What types of connections do PSTs make?
2. How did the types of connections PSTs make change throughout the study?

Methodology and Data Analysis

Data for this manuscript were collected at an urban university in the Southeastern United States. The sample was composed of 18 third- and fourth-year students who were completing their bachelor's degree in education. The study was conducted in a mathematics education content and methods class required for all early childhood, elementary, and special education majors. This semester long course met twice a week for one hour and 15 minutes for 16 weeks. The first two authors were the instructors of the course and the third author had extensive experience teaching that course. All three authors met weekly to discuss students' writing and create problems to further PSTs' understanding of making connections.

To guide each lesson, we modeled Smith and Stein's (2011) five practices. First, the instructors met outside of class and created an elementary mathematics task and listed anticipated solution strategies informed by the literature (Levi & Empson, 2011; Van de Walle, Karp, & Bay-Williams, 2015). Then, in class, we followed two approaches to assess PSTs' connection making. The approach included presenting PSTs with the task, asking them to solve it on their own, and asking them to share their strategies in small groups. At this point, we modeled the practices of monitoring, selecting, and sequencing. Frequently we chose three students to share their solution strategies with the entire class. Once the chosen students shared their strategies, we led a discussion where we asked the following recurring questions:

1. What big mathematical idea can you address by using this problem?
2. What connections can you make between the strategies to highlight the big idea?

In alternate lessons, we provided PSTs with possible solution strategies and then asked them the two questions above. In this scenario, we placed the PSTs in the role of the teacher by asking them to decide how they would proceed in a classroom where the given answers appeared (Grossman, Hammerness, & McDonald, 2009). We asked the two questions above to help PSTs clarify their thinking and to help us assess their thinking. The first question was necessary to understand the learning goal PSTs were looking to emphasize as well as to help demonstrate the importance of keeping the goal of the lesson at the forefront. The responses to the second question uncovered the types of connections PSTs were making and was designed to narrow PSTs answers to focus on mathematical connections between strategies. This writing was designed to help PSTs clarify, and then, formalize their thinking about the mathematical connections in the solutions presented. The writing was also used by the authors as formative assessment providing information about PSTs understanding and thinking about connections.

Our data consisted of nine responses to the two recurring questions above for each PST in our sample. Each week, PSTs engaged in a problem-solving activity and completed the writing assessment. After each activity, we collected students' writing, removed any identifying information, and made photocopies to use during our weekly meetings. We returned the original copies to PSTs with feedback about the connections they discussed. During our weekly meetings, we read PSTs responses, identified common codes, and determined themes describing the types of connections PSTs were making (Marshall & Rossman, 2006; Taylor & Bogdan,1998). The information obtained during these meetings informed the development of the activity used in the following week's lesson. We continued this process throughout the semester, analyzing each new set of data, updating and refining our list of codes, and using this information as we designed the curriculum.

As we read through PSTs' responses to the first activity, we used content analysis, which focuses on identifying concepts within texts, and created a list of common phrases we observed throughout their writing (Carley, 1993). Then, two of the researchers coded the assignment using this list while looking for emergent codes (Taylor & Bogdan 1998). For example, in response to question two, a PST stated, "I would also explain to them how 6/9 simplifies to 2/3 so they can understand that both fractions are still equivalent (1). I would even take one of the drawings from Ana's picture and one from Ben's and shade in 2/3 and 6/9 so they can visually see they're equal (2)". We coded the first statement (1) as *describing the big idea* because the PST described equivalence and statement (2) as *multiple representations* because the PST pointed out that two different representations were equivalent to each other. Throughout the process, we also compared our coding to combine similar codes, delete infrequent codes, and write definitions for each code. For example, the code *preferred learning styles* and *variety* were combined to create the code *various strategies.* This process was repeated several times while concurrently informing our coding by reading research in mathematics education. When the codes were finalized, the data were coded again in their entirety. All disagreements were discussed at length until we arrived at the same conclusion. For example, during the first iteration, learning styles and variety emerged as separate codes. After further discussion, we concluded that in both codes PSTs were mentioning different strategies without connecting them, thus we combined these to create the code of *various strategies.*

Then, we carefully examined all codes to search for themes. When several codes described the same component within the data, we combined them to create an overarching theme. For example, the codes *same answer, naming a concept,* and *various strategies* were combined to create the theme *superficial knowledge connections.* Finally, each theme and code were defined (see Table 1). These themes are representative of the types of connections made by PSTs.

For our analysis of the first research question investigating the types of connections PSTs made, we analyzed the assessments associated with all 9 problem-solving activities. For the analysis of the second research question where we sought to investigate how the types of connections PSTs made changed throughout the study, we selected the assessments from three activities, one that took place at the beginning, middle, and end of the semester. We will briefly describe the problems that guided the three activities (see the Appendix for the full problems. The first activity involved solving the problem 70 minus 59. PSTs were presented with strategies involv-

Table 1

Codes with their descriptions and examples.

Code	Description	Student Example
Superficial Knowledge Connections		
Same answer	Emphasis on the numerical result being the same.	Sandra, Alice and Milagros used differing strategies and got the same answer.
Naming a concept	Stating the use of a concept without showing understanding.	The connections that I would address would be place value and how all of the examples kept place value in mind.
Various strategies	Mentioning a variety of solution strategies without connecting them.	Milagros does the problem very traditionally where as Alice breaks her problem down.
Procedural Knowledge Connections		
Step by step	Describing step by step the process used to get the answer.	I could also ask the students how many 3's go into 6, they would get 2. Then, ask them how many 3's go into 9 and they would get 3, therefore ⅔ = 6/9.
Operation used	Pointing out the same operation was used in both or more strategies.	All use subtracting in one form or another to get to the final answer.
Conceptual Knowledge Connections		
Describing the big idea	Students' understanding of general rules, facts, and definitions and using these appropriately to describe the big idea.	The big idea could be the commutative property. Connection is that all the numbers are grouped differently showing that you can switch the order in which its multiplied. For example, 6 x 3 to 3 x 6.
Knowing why a procedure works	Students' understanding of why a specific procedure works and the ability to apply it to solve math problems.	Ben's strategy is a more expanded version of Cal's because he drew out the 6 subs technically and divided it into 9 each time. Cal, on the other hand, automatically knew that he would have to use the drawing that he made 6 times for the 6 subs, so he skipped the addition and multiplied instead.
Multiple representations	Pointing out how the same idea is represented in two different representations.	If you cut the pieces from Ana's strategy, you can see the 9 pieces from Cal's strategy. They are equivalent.

ing the traditional algorithm, the split method, and compensation (Van de Walle, John, Karp, & Bay-Williams, 2015). To guide the activity used in the middle of the course, PSTs were asked to solve the problem six times three in three different ways: grouping using manipulatives, an array, and using a number line. The last activity introduced an equal sharing problem that stated, "At a restaurant, the waiter brings six sub sandwiches for nine children to share so everyone gets the same amount. How much will each child have?" Three student strategies were presented, each with a distinct drawing showing how the sandwiches could be shared equally among the nine children. After each activity was completed, PSTs were given a writing assessment with our standard questions:

1. What is the big mathematical idea you would like your students to understand?
2. What connections would you address that would highlight the big idea?

Results

To address the first research question examining the types of knowledge connections made by PSTs, we looked at PSTs' responses across the nine assessments used during the course. From our data, we identified three types of connections made by PSTs: superficial knowledge connections, procedural knowledge connections, and conceptual knowledge connections (see Table 1).

Superficial knowledge connections

Connections that use superficial knowledge focus on shallow features of the strategies presented. Within this theme, there are three codes that clarify what we considered a superficial connection. The codes we identified were called: same answer, naming a concept, and identifying various strategies.

Same answer. PSTs identified a connection by describing that multiple strategies led to the same answer. For example, one PST connected students' strategies by saying, "Sandra, Alice, and Milagros used differing strategies and got the same answer." While we believe it is important for PSTs to understand that multiple strategies can produce the same result, we identified this connection as superficial because it does not build upon students' procedural or conceptual understanding.

Naming a concept. PSTs connected strategies by simply naming a mathematical concept used during the solution process. In the quotation below, a PST described the connection as being about *place value*—"the connections that I would address would be place value and how all of the examples kept place value in mind." Although building understanding about place value was the learning goal associated with the mathematics problem, in this instance the PST did not explain how the solution strategies could be used to build understanding about place value, instead the PST only mentioned that place value was evident in all the strategies used. For this reason, we considered a connection such as this one, which names the concept used in the strategies, as a superficial knowledge connection.

Identifying various strategies. PSTs identified that various strategies were used to solve the same problem. In this type of connection, PSTs did not find a relationship between the strategies; they merely mentioned that various strategies were used. For example, "In Alice and Milagros problem, they are both subtracting. However, Milagros does the problem very traditionally whereas Alice breaks her problem down." Here, the PST noticed the problem was solved using different strategies, but the fact that there was no attempt to make a connection between these strategies led us to define this as a superficial knowledge connection. Thus, focusing on superficial knowledge connections will rarely bring forth important discussions about mathematical concepts.

Procedural knowledge connections

The second type of connections we identified were procedural knowledge connections. In this type of connection, PSTs focused their attention on mathematical procedures. We identified two codes for procedural connections, describing a step-by-step solution path and stating the operation that was used to solve the problem.

Step-by-step solution. One PST described the following connection: "strategy 1 got 6/9 and strategy 2 got 2/3. When you simplify strategy 1, you get 2/3 because 6 and 9 are divisible by 3 so 6/3 = 2 and 9/3 = 3 so you are left with 2/3." The PST made a connection between strategies by emphasizing a procedure, simplifying fractions by dividing both parts of the fraction by the same number. We identified this as a procedural knowledge connection because it focused on a step-by-step process for solving the problem. While the connection shown above does help students think about a procedure for simplifying fractions, it falls short of helping students build understanding about why the procedure works.

Stating the operation. PSTs also made connections by stating operations that were used to solve a problem. For example, to connect all three strategies in one activity, one PST said, “Sandra’s strategy is drawing blocks, Alice’s strategy is grouping and subtracting, and Milagros’ strategy is traditional vertical subtracting.” Here, a connection was made by identifying the different procedures used in each strategy to illustrate subtraction. However, the PST simply described the different approaches of subtraction and missed an opportunity to connect student strategies by highlighting the use of place value and the properties of operations needed to subtract (CCSSO, 2010).

Conceptual knowledge connections

The last type of connection we identified involved making conceptual knowledge connections. In this theme, we identified the following codes to help demonstrate what we mean by conceptual knowledge connections: describing the big idea, knowing why a procedure works, and using multiple representations. In each of these codes, PSTs described a connection in such a way we believed the connection would assist students in building conceptual understanding of mathematics.

Describing the big idea. The code describing the big idea refers to examples where the PST created a connection that focused on students’ understanding of general mathematical rules, facts, and/or definitions. We coded the following connection as conceptual and noted that it is an example of describing the big idea, “I would have them shade the number of 9’s in Ana’s answer on the 1st sandwich and shade the number of 9’s from Ben’s answer on the second sandwich to show that 2/3 and 6/9 are equivalent. I would have the student point out the six sets of 1/9 from Cal’s strategy in Ben’s answer to show how it is also equivalent to 2/3.” This PST demonstrated an understanding of the big mathematical idea of equivalency through her assertion that 2/3 and 6/9 are both accurate answers. More specifically, this PST used students’ visual fraction models to highlight fraction equivalence, where two fractions with different number and size of parts have the same numerical value or are on the same point on a number line (CCSSO, 2010).

Knowing why a procedure works. An example of this was when a PST made the following connection, “Ben added 1/9, six times to give him 6/9 and Cal represented 1/9 in his drawing and then multiplied it by six to get 6/9. It can be said that Cal and Ben used the same method because although Cal multiplied, multiplication is a form of repeated addition, which is what Ben used.” In this case, the PST made a connection by describing the process of multiplication and relating it to repeated addition. This type of conceptual knowledge involves relating concepts to specific procedures and showing understanding for why certain procedures work for particular problems (Crooks & Alibabi, 2014).

Using multiple representations. The final type of conceptual knowledge connection involved multiple representations and was shown when PSTs pointed out the same idea in two different representations. An example of this type of connection was, “I would have the students point out the 6 sets of 1/9 from Cal’s strategy in Ben’s answer to show how it is also equivalent to 2/3.” In a picture included with the response, this PST showed how the six, one-ninths in Cal’s strategy could fit within the six, one-ninths in Ben’s strategy and then shaded two columns in Ben’s strategy to demonstrate how this represented two-thirds. Thus, this conceptual knowledge connection showed an understanding of equivalency by comparing two distinct representations.

Furthermore, we also noticed PSTs sometimes made more than one conceptual knowledge connection within the same response. One PST observed, “the big idea is that *6 x 3* and *3 x 6* is the same thing (commutative property). With all the strategies it can be written in both ways and once they’re solved it’s the same thing. For example: ||| ||| ||| ||| ||| |||=|||||| |||||| ||||||.” Here the PST started by describing the big idea and continued by explaining multiple representations of the same concept. Then, the PST simplified her explanation by using sticks to draw six groups of three and equating them to three groups of six. This demonstrates an understanding of the commutative property in relation to the connection made between the strategies.

As these three types of connections demonstrate, conceptual knowledge connections address ideas that go beyond specific procedures for solving tasks. By addressing the concepts underlying the mathematical strategies used to solve the problem, conceptual connections have the ability to extend mathematical conversations and assist in the development of important mathematical ideas.

Next, we turn our attention to the second research question: How did the types of connections made by PSTs change throughout the study? To investigate this question, we used results from the three assessments that took place during the beginning, middle, and end of the semester. The purpose of this analysis was not to look at changes in individual PSTs, but to look at the sample as a whole to determine which types of connections were most common during which stage of the course. This is helpful in understanding how PSTs' connections changed from the beginning to the end of the course and provides us with valuable information about whether our instructional intervention led PSTs to make more conceptual connections as the course progressed.

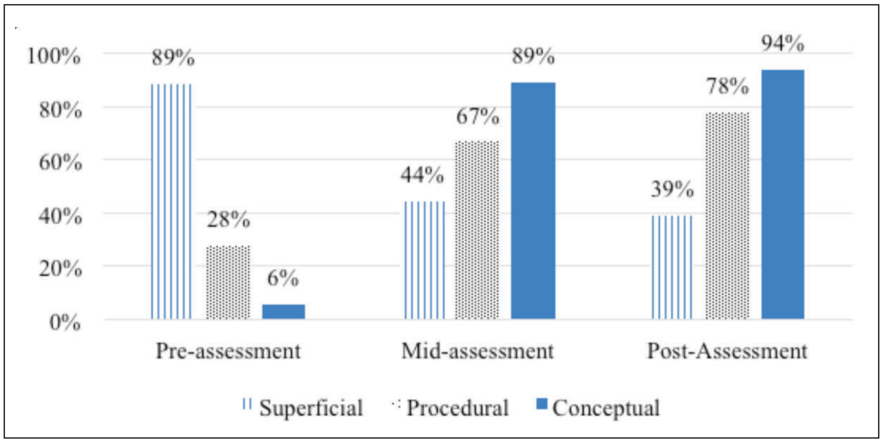


Figure 1. Proportion of students who made each type of connection across activities.

In Figure 2, we can see the proportion of PSTs who made each type of connection by assessment. For example, the 89% above column one on the pre-assessment means that 89% of PSTs in the sample made at least one superficial knowledge connection on the pre-assessment. Preservice teachers were able to make multiple connections on the same assessment. Thus, a PST who made a superficial knowledge connection may have also made a procedural knowledge connection and/or a conceptual knowledge connection on the same assessment.

Comparing the proportion of superficial knowledge connections across the three activities, we see that the amount of this type of connections decreased throughout the study. At the beginning of the course, 89% of PSTs made superficial knowledge connections, while at the end of the course only 39% made superficial knowledge connections. The high proportion of PSTs making superficial knowledge connections on the first activity was coupled with a relatively low proportion of PSTs making either procedural or conceptual knowledge connections. This suggests that PSTs' knowledge at the beginning of the course was focused on superficial types of connections. The results at the end of the course looked very different. The last activity showed that PSTs' knowledge about connections focused less on superficial knowledge and more on procedural and conceptual knowledge connections. In fact, by the end of the course, 94% of PSTs identified at least one conceptual knowledge connection, up from 6% at the beginning of the course.

When examining the connections made on the last assessment, we noticed PSTs were understanding how to use the big idea and the strategies together to build conceptual understanding. For example, one PST said, "I would also explain to them how 6/9 simplifies to 2/3 so they can understand that both fractions are still equivalent. I would even take one of the drawings from Ana's picture and one from Ben's and shade in 2/3 and 6/9 so they can visually see they're equal." Here, we see how this student first makes a connection when she describes the big idea of equivalence using the strategies provided, and then, she makes another connection as she continues to clarify the big idea by using its definition to demonstrate with multiple representations how two fractions are equivalent.

Discussion and Implications for Further Research

When considering the ability of a connection to promote mathematical understanding, focusing exclusively on superficial knowledge connections may provide students with opportunities to see obvious similarities between strategies, such as having the same answer, but it also hinders the direction of the whole classroom discussion when thinking about big mathematical ideas. Procedural knowledge connections are a salient part of a classroom discussion because they help students develop instrumental understanding; however, these connections do not encourage students to move beyond describing a procedure to understand why it works. Conceptual knowledge connections, on the other hand, promote rich discussions that lead to the big idea of the lesson and makes the mathematical learning goal visible to the entire class. Conceptual connections build relational understanding as teachers work to help students generalize ideas and identify them across multiple mathematical representations.

Ensuring PSTs are able to make connections and understand the roles connections make in learning mathematics is necessary, but PSTs must also learn how to draw connections out of their students. Throughout the course, we modeled the behavior of a teacher who is drawing out connections from his or her students; however, we never explicitly worked with PSTs to help them acquire this skill. We consider this a shortcoming in our intervention and plan on including this in the next iteration of our teaching. As a starting point, we note several questions that PSTs could be instructed to use when working with their own students that we believe assist in getting students to think about connections. These are:

- What makes you say that?
- What are some similarities (or differences) you see between the solution strategies?
- How do these similarities and differences help you understand the big idea?
- How are you able to see X represented in solution Y?
- How does X representation help you understand why procedure Y works?
- We also note examples of specific questions we asked during the activities that assisted PSTs in making mathematical connections. For example, when discussing the pre-assessment, after PSTs had a chance to write about the connections they saw, we used the following questions to push their thinking further:
- Where do you see the use of place value in strategy one? Where do you see place value in strategy 2? How would you use this to help your students understand the big idea?
- In this strategy the student is borrowing from the tens place to subtract. Is this process happening in another strategy?

When discussing the activity from the middle of the semester, we inquired:

- Where do you see the row of six represented in any of the other strategies?
- How does the array show the addition, 6 + 6 + 6?

Finally, when discussing the last activity, we asked:

- Are 2/3 and 6/9 equivalent? Use the strategies to defend your answer.
- How can we use the strategies to show fraction equivalence?

Subsequently, we posit that being transparent about the questions we use in class and why will be helpful in the future.

As educators, this action research project was a practical way to improve our practice, while simultaneously creating knowledge that is valuable to other educators (Somekh & Zeichner, 2009). Because we had a specific goal in mind, that of making connections, we were able to concentrate on a crucial piece of developing understanding in mathematics. We also saw the value in this type of research and were eager to share it with other educators. This is one of the many purposes of action research, sharing useful information with peers in similar positions (Somekh & Zeichner, 2009). Although this study was conducted with elementary PSTs, research shows secondary PSTs compartmentalize mathematical ideas and have difficulty making connections between solution strategies (Even, 1993; Moon, Brenner, Jacob & Okamoto, 2013). It would be interesting to implement this intervention in a secondary mathematics content and methods course to see if similar types of connections emerge. Hence, we understand that more iterations of this action research project need to happen to further understand the types of connections PSTs make and how they improve over time. Nevertheless, this study adds to the body of literature on innovative practices in mathematics education and in action research.

Appendix

Pre-Assessment

You present your second grade class with the problem 70-59. Your students solve it in the following ways.

Traditional Algorithm	*Compensation Method*	*Split Method*
60	70 – 59	70 – 59 =
70	70 – 60 = 10	70 – 50 – 9 =
–79	+1	20 – 9 = 11
11	11	

a. What big mathematical idea can you address by using this problem?

b. What connections can you make between the strategies to highlight the big idea?

Mid-Assessment

Students present three ways to solve the problem 6 x 3: grouping using manipulatives, array, and number line. Solve 6 x 3 using these three strategies, then use these strategies to answer the following questions.

a. What big mathematical idea can you address by using this problem?

b. What connections can you make between the strategies to highlight the big idea?

Post-Assessment

Imagine you are a 3rd grade teacher. In your class, you are teaching equal sharing using the following problem:

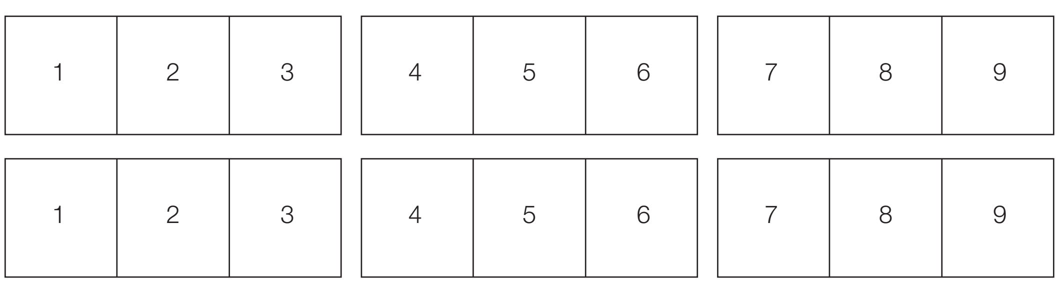
At a restaurant, the waiter brings 6 sub sandwiches for 9 children to share so everyone gets the same amount. How much will each child have?

Three of your students' strategies are below. Use these strategies to answer the following questions.

a. What big mathematical idea can you address by using this problem?

b. What connections can you make between the strategies to highlight the big idea?

Ana's Strategy:



Each number represents a child.
Each child gets 1/3 + 1/3 = 2/3 of a sub sandwich.

Bens's strategy:

1	2	3
4	5	6
7	8	9

1	2	3
4	5	6
7	8	9

1	2	3
4	5	6
7	8	9

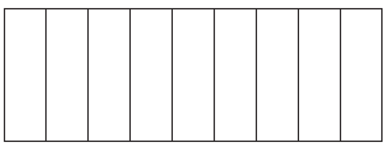
1	2	3
4	5	6
7	8	9

1	2	3
4	5	6
7	8	9

1	2	3
4	5	6
7	8	9

1/9 + 1/9 + 1/9 + 1/9 + 1/9 + 1/9 = 6/9
6/9 of a sub each.

Cal's strategy:



Each sub split into 1/9 pieces and each child will get 1/9 of each sub.
There are 6 subs so each child will get 1/9 x 6 = 6/9

References

Association of Mathematics Teacher Educators [AMTE]. (2017). *Standards for Preparing Teachers of Mathematics.* Retrieved from amte.net/standards.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The elementary school journal, 93*(4), 373–397.

Bingolbali, E., & Cusknn, M. (2016). A proposed conceptual framework for enhancing the use of making connections skill in mathematics teaching. *Egitim Ve Bilim-Education and Science, 41*(183), 233–249.

Caine, R.N., & Caine, G. (1991). *Making connections: Teaching and the human brain.* Menlo Park, CA: Addison-Wesley.

Carley, K. (1993). Coding choices for textual analysis: A comparison of content analysis and map analysis. *Sociological Methodology, 23,* 75–126.

Chapin, S. H., O'Connor, C., & Anderson, N. C. (2009). *Classroom discussions: Using math talk to help students learn, grades k-6.* Math Solutions.

Council of Chief State School Officers [CCSSO]. (2010). Common Core State Standards for Mathematics. Washington, DC: CCSSO.

Crooks, N. M., & Alibali, M. W. (2014). Defining and measuring conceptual knowledge in mathematics. *Developmental Review, 34,* 344–377.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education, 24*(2), 94–116.

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, Re-imagining teacher education. *Teachers and Teaching: Theory and Practice, 15*(2), 273–289.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In Hiebert, J. (ed.), *Conceptual and procedural knowledge: The case of mathematics.* New Jersey: Lawrence Erlbaum Associates Inc.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., … Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding.* Portsmouth, NH: Heinemann.

Lampert, M. (2001). *Teaching problems and the problems of teaching.* Yale University Press.

Levi, L., & Empson, S. B. (2011). *Extending children's mathematics: Fractions and decimals.* Portsmouth: Heinemann.

Marshall, C., & Rossman, G. B. (2006). *Designing qualitative research. T*housand Oaks, CA: Sage Publications.

Mhlolo, M.K., Venkat, H., & Schäfer, M. (2012). The nature and quality of the mathematical connections teachers make. *Pythagoras, 33*(1), 1–9.

Moon, K., Brenner, M. E., Jacob, B., & Okamoto, Y. (2013). Prospective secondary mathematics teachers' understanding and cognitive difficulties in making connections among representations. *Mathematical Thinking & Learning, 15*(3), 201–227.

National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standards for teaching mathematics.* Reston, VA: Author.

Patthey, G. G., & Thomas-Spiegel, J. (2013). Action research for instructional improvement: The bad, the ugly, and the good. *Educational Action Research, 21*(4), 468–484.

Rittle-Johnson, B., Siegler, R.S., & Alibali, M.W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*(2), 346–362.

Schoenfeld, A. S. (1998). Toward a theory of teaching-in-context. *Issues in Education, 4*(1), 1–95.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teacher, 77*, 20–26.

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematical discussions.* Reston, VA: National Council of Teachers of Mathematics.

Somekh, B., & Zeichner, K. (2009). Action research for educational reform: Remodeling action research theories and practices in local contexts. *Educational Action Research, 17*(1), 5–21.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313–340.

Taylor, S.J., & Bogdan, R. (1998). Introduction to Qualitative Research Methods. Wiley and Sons: Hoboken, NJ.

Tchoshanov, M. A. (2011). Relationship between teacher knowledge of concepts and connections, teaching practice, and student achievement in middle grades mathematics. *Educational Studies in Mathematics, 76*(2), 141–164.

Van de Walle, J. A., Kapp, K. S., & Bay-Williams, J. M. (2015). Elementary and Middle School Mathematics: Teaching Mathematics Developmentally. Ninth Edition. Boston, MA: Pearson Education Inc.

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Doctoral Preparation in Mathematics Education — Time for Research and a Widespread Professional Conversation

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ABSTRACT Doctoral programs in mathematics education were established more than a century ago in the United States. From 2010-2014 over 120 different institutions graduated at least one doctorate in mathematics education. There has been limited research reported on the nature of doctoral programs in mathematics education and/or their doctoral graduates. This paper provides a synthesis of research findings related to doctoral preparation in mathematics education that is accompanied by a reflection on the findings and suggestions for future research. The intent of our paper is to provide a rallying call for more widespread and coordinated research on doctoral programs in mathematics education in order to strengthen the quality of doctoral preparation for the next generation of mathematics educators.

KEYWORDS *accreditation, core knowledge, doctoral preparation, mathematics education*

Doctoral programs in mathematics education were established more than a century ago but the majority of programs began to evolve about 60 years ago (Donoghue, 2001). Today, U.S. programs collectively graduate about 130 students each year (Reys & Reys, 2016). These graduates have an important and widespread influence on the field as they do much of the research reported in peer reviewed journals, have major responsibility for preparing the next generation of K-12 teachers, and serve in many leadership roles in professional organizations that represent mathematics educators. Given the influence of graduates of mathematics education doctoral programs and the community of mathematics educators who oversee these programs, it is surprising that ongoing collaboration and/or conversation about the improvement of these programs is rare. This commentary is intended to stimulate more conversation.

About 70 different institutions graduate at least one doctorate in mathematics education annually. However, a graduate of one program is likely to have had very different experiences/preparation than a doctoral student at another institution. Research suggests that even doctoral programs at peer institutions differ significantly in a range of factors, including the number and type of courses and/or internships required, the nature and quantity of mathematics education courses offered, the extent to which mathematics content is a focus, the number of mathematics education faculty and/or doctoral students in the program, and the length of the program (McIntosh & Crosswhite, 1973; Soonabend, 1981; Reys, Glasgow, Ragan, & Simms, 2001; Reys, Glasgow, Teuscher, & Nevels, 2008). Upon completing a doctorate in mathematics education the graduate chooses among many different career paths, but the majority pursue a career in higher education (Glasgow, 2000).

Do doctoral graduates in mathematics education share a core base of knowledge? If so, what constitutes the core? What characterizes strong doctoral programs in mathematics education? Do certain doctoral programs in mathematics education better serve students with specific career goals (such as preparing them for collegiate teaching of mathematics or to conduct research)? Where are the highly regarded doctoral programs in mathematics education? These questions are rarely addressed or discussed in the mathematics education community. However, there are data available that sheds light on how current faculty members perceive particularly strong doctoral programs in peer institutions (Reys, et al., 2008; Reys, Reys, Shih, & Safi, 2019).

According to the Carnegie Foundation, one of the purposes of doctoral study is to prepare stewards of the discipline (Golde & Walker, 2006; Reys & Dossey, 2008). In this commentary, we highlight some research from surveys of doctoral graduates in mathematics education and active faculty members in doctoral programs in mathematics education. Our goals are to inform the reader about the current status of doctoral preparation; encourage collaboration, discussion, and regular examination of doctoral programs in mathematics education; and stimulate more research focused on doctoral preparation in mathematics education.

Foundational Research Related to Doctoral Programs

There has been little published research on doctoral preparation in mathematics education (Kilpatrick & Spangler, 2016; Reys, 2017). For example, in a review of five decades of mathematics education research published in the *Journal for Research in Mathematics Education* and *Educational Studies of Mathematics* there was not one citation that mentioned doctoral preparation in mathematics education (Inglis & Foster, 2018). There are several early surveys of doctoral programs available thru ERIC (McIntosh & Crosswhite, 1973; Soonabend, 1981), and then two surveys (Reys et al., 2001; Reys, et al., 2008) that were done in conjunction with national conferences on doctoral programs in mathematics education (Reys & Kilpatrick, 2001; Reys & Dossey, 2008). Since that time there has been a survey of doctoral graduates in mathematics education (Shih, Reys, & Engledowl, 2016; Shih, Reys, Reys, & Engledowl, 2019), a survey of faculty members actively involved in doctoral programs in mathematics education (Reys, et al., 2019) and some periodic reviews of job shortages in the field (Reys, 2002; Reys, Reys, & Estapa, 2013) and production of doctoral graduates (Reys & Reys, 2016). A brief summary (a baker's dozen) of findings from those studies includes:

1. The majority of doctoral programs in mathematics education are in the college/school of education, some institutions offer a doctorate in mathematics education in both the college/school of education and mathematics department, and a few institutions offer their doctorate in mathematics education exclusively in the mathematics department.
2. During the last 50 years, doctoral programs in mathematics education have been established or grown at some institutions and declined or eliminated at others. Overall, the number of different institutions graduating at least one doctorate in mathematics education has increased from about 37 during 1960-1962 to about 130 during 2010-2014.
3. The total number of doctorates in mathematics education averaged about 50 during the 1960s and about 130 during the period 2010-2014. Whereas the majority of doctoral graduates in mathematics education in the 1960s were male (about 80%), currently about two-thirds of the doctoral graduates are female.
4. Most doctoral programs in mathematics education are very small. Few institutions graduate a doctorate in mathematics education annually, and only two institutions (Teachers College and University of Georgia) have averaged at least 5 or more graduates annually since 2000.
5. There was an acute shortage of doctorates in mathematics education for more than two decades (1990-2010). There now seems to be an equilibrium between jobs available in mathematics education and new graduates. The exception is the continued shortage of doctorates in mathematics education in mathematics departments of private and regional institutions.
6. The number of faculty members actively involved in doctoral programs in mathematics education varies from 1 in some institutions to more than 10 in other institutions, with a mode of 4 faculty members. Over one-third of the institutions graduating doctorates in mathematics education have 3 or fewer faculty members in mathematics education.
7. The majority (nearly two-thirds) of faculty members working in doctoral programs are female, and this parallels the percent of new female doctorates in mathematics education that graduated during the last 15 to 20 years.
8. About one-fifth of the faculty members in doctoral programs in mathematics education have no K-12 teaching experience and most have three or fewer years of K-12 teaching experience.
9. The number of graduate level mathematics courses required for completion of a doctorate in mathematics education varies among institutions from none to at least a master level degree in mathematics.
10. The number of graduate courses specifically focused on mathematics education available at institutions offering a doctorate range from 0 to more than 10 with 4 to 6 courses being most typical.

11. Doctoral graduates in mathematics education were generally very positive about their doctoral program. The two areas most often cited in need of strengthening were opportunities to gain first-hand experience in preparing proposals for funding and sustained involvement in active research projects.
12. About one-third of faculty members working in doctoral programs reported regularly soliciting feedback about the program from their graduates, while about 15% reported they do not have a system for seeking feedback. At least one faculty member from over 90% of the institutions graduating the most doctorates in mathematics education was familiar with the *Principles to Guide the Design and Implementation of Doctoral Programs in Mathematics Education* (Association of Mathematics Teacher Educators [AMTE], 2003).
13. About one-half of faculty members reported carefully reviewing their doctoral program in mathematics education within the last two years, and about one-quarter of the faculty members indicated they did not know when their doctoral program had been last reviewed.

Some reflections on these findings

More institutions and smaller programs. For the last 50 years there has been an increase in the number of different institutions offering a doctorate in mathematics education. Yet most doctoral programs in mathematics education are small and expensive to operate. That is, many smaller programs cannot afford to offer specific courses unique to mathematics education. Instead, they offer general education courses on curriculum, history, and the psychology of learning that serve graduates from multiple disciplines, and they may offer independent study courses focusing on mathematics education issues (Bay, 2001). These faculty members likely have major responsibilities in their undergraduate programs as well, so it raises questions about the amount of time they have to engage in research, mentor doctoral students through all phases of their program, in addition to other ongoing responsibilities (committee work, scholarship, proposal writing, etc.) that are typically expected of faculty members at doctoral granting institutions (Foley, 2014). This situation of small graduate programs reflecting few students and faculty members prompted Levine (2007) to comment that we have "too many under resourced doctoral programs for the preparation of education scholars" (p. 60). Levine was particularly critical of the research mentoring and preparation provided to doctoral students in programs with faculty members that were not engaged in scholarly research. Levine went on to "recommend the establishment of high and clearly defined standards for education research and doctoral preparation in research; close doctoral programs that do not meet those standards" (p.75). We concur that limited resources pose severe challenges for establishing and maintaining a high-quality doctoral program in mathematics education.

Intellectual communities. To strengthen doctoral programs, some have called for the establishment of intellectual communities (Golde, 2008; Hiebert, Lambdin, & Williams, 2008). An intellectual community is formed around domains of knowledge that involve active faculty participation and leadership that provide models, mentoring, and apprenticeships for doctoral students. Examples include intellectual communities focused on teaching, curriculum, or equity/diversity as were fostered through the NSF Centers for Learning and Teaching initiative in the first decade of this century[1]. Some institutions with a large number of faculty members may have several different intellectual communities operating simultaneously. While the minimum number of people needed to form intellectual communities may vary, it would certainly be a challenge for such communities to exist with three or fewer faculty members.

K-12 teaching experience. Arguments have been made that doctoral candidates in mathematics education should have PreK–12 teaching experience prior to entering a doctoral program (Reys, 2018a; AMTE, 2003). For example, PreK–12 classroom teaching experience provides essential grounding and ensures first-hand experience with PreK–12 students working in school environments. Such PreK–12 teaching experience provides valuable credibility for mathematics educators working with future and in-service teachers.

While the majority of people entering doctoral programs in mathematics education have some K-12 teaching experience, about 20% of current mathematics education faculty in higher education have no K-12 teaching experience. This lack of K-12 teaching experience may put the doctoral student at a disadvantage in

[1] Appalachian Collaborative Center for Learning, Assessment, and Instruction in Mathematics (ACCLAIM), Center for Mathematics Education of Latinos/as (CEMELA), Center for Mathematics in America's Cities (MetroMath), Center for Proficiency in Teaching Mathematics (CPTM), Center for the Study of Mathematics Curriculum (CSMC), Diversity In Mathematics Education (DIME), and Mid-Atlantic Center for Mathematics Teaching and Learning (MAC-MTL).

some courses that call upon relating content to teaching and learning mathematics. It may also prove troublesome in establishing credibility with college students in teaching methods courses and perhaps even supervising student teachers in schools. It may also eliminate some job opportunities, as some states, such as Alabama and Minnesota, require faculty members supervising field experience and/or teaching pre-service teachers to have had K-12 teaching experience.

A focus on mathematics content knowledge. What is an appropriate level of mathematics content knowledge for graduates of doctoral programs in mathematics education? This was a major area of attention at the first national conference on doctoral programs in mathematics education (Reys & Kilpatrick, 2001). There was general agreement that doctoral graduates in mathematics education should have foundational knowledge of mathematics, although there was not consensus on the extent or nature of that knowledge. Mathematics content was one of the common core recommendations reported in the *Principles to Guide the Design and Implementation of Doctoral Programs in Mathematics Education* (AMTE, 2003). Some details have been spelled out about mathematics content background with the amount of mathematical knowledge being a function of whether the doctoral student is focusing on elementary, secondary or collegiate levels (Dossey and Lappan, 2001). Nearly twenty years after this conference the amount of foundational mathematics content required for a doctorate in mathematics education continues to vary across, and oftentimes within, institutions. Most institutions require some graduate level mathematics for completion of a doctorate, but this requirement may depend on the major advisor and doctoral committee. While the mathematics content required in the colleges/schools of education may vary, doctorates in mathematics education awarded by mathematics departments typically require at least the equivalent of a master's degree in mathematics.

Improving the system. Doctoral programs in mathematics education have been called a 'complex system' (Hiebert, Kilpatrick, & Lindquist, 2001). Improving complex systems cannot be done easily or quickly, and one framework for improving complex systems consists of four steps: assess current conditions; clarify goals; develop strategies for moving from current conditions to goals; and document and share information about the effects of improvement strategies (Hiebert, et al., 2008). While specific goals of institutional programs may vary, the *Principles to Guide the Design and Implementation of Doctoral Programs in Mathematics Education* (AMTE, 2003) provides a blueprint. Faculty members at most institutions are familiar with this document, so it might serve as a helpful guide. Obtaining feedback from doctoral graduates to help shape and strengthen a program reflects the first step toward improvement. Yet less than one-third of the institutions reported gathering feedback from graduates annually, and over 15% of the faculty members said they have never gathered such feedback. It has been argued that such regular feedback would provide valuable information to integrate into programmatic reviews and make progress toward future steps in improving doctoral programs (Reys & Reys, 2017). Our hope is that this commentary will encourage all faculty members involved in doctoral programs in mathematics education to become involved in shaping requests for feedback from doctoral graduates, agree upon the frequency of this effort, and periodically contribute to a careful review of their doctoral program.

Accreditation. Is it time for the field to consider establishing an accreditation system for doctoral programs in mathematics education? Program accreditation is widespread in many areas of higher education. It was a topic discussed at the second national conference that stimulated thoughtful discussion on both sides (Lappan, Newton, & Teuscher, 2008). It was agreed that an accreditation process would require guidelines and standards that could be used to develop and assess the quality of doctoral programs in mathematics education, and to better define what is meant by a doctorate in mathematics education. Furthermore, it was agreed that external reviews from an accreditation would encourage more regular self-examination and thoughtful discussions by faculty members leading the doctoral program. Music education provides an existence proof that accreditation of doctoral programs in an education discipline can be successfully carried out (Reys, 2018b).

The ultimate goal of reviewing and accrediting doctoral programs in mathematics education would be to strengthen doctoral preparation. The accreditation process should be constructive. It should also provide a pathway to help new doctoral programs become established as well as strengthen and help keep established doctoral programs dynamic. An accreditation report would summarize program strengths and weaknesses and this feedback could be used by faculty members to develop an action plan going forward. Such information could be used by faculty members to leverage support from administrators to strengthen their doctoral program. It might also be used to denote accreditation status for their doctoral program in mathematics education, thereby attracting more doctoral students.

Some possible directions for future research

Research related to doctoral preparation in mathematics education has been limited and rarely reported in peer-reviewed journals (Kilpatrick & Spangler, 2016; Reys, 2017). Yet, if the mathematics education community is to grow and become stronger, research is needed on many fronts related to doctoral preparation in the field. Some possible directions for future research are offered here:

- Identify institutions that offer the option of a Ph.D. or an Ed.D. in mathematics education. What program requirements are the same? How are they different? Do career paths for recipients of these degrees differ? If so, in what ways?
- Examine the number of applications, acceptance rate, attrition rate, and graduation rate of students entering doctoral programs in mathematics education by institution. Are the attrition-graduation rates different across institutions? If so, why?
- Identify syllabus/content descriptions of graduate level courses for doctoral students in mathematics education. How similar/different are courses focusing on similar topics, such as mathematics curriculum or learning mathematics? Is there a required minimum type or number of these courses that all doctoral graduates in mathematics education must complete? Is there a rationale for this requirement?
- Collect and analyze required readings, courses, internships, and other experiences of current doctoral programs. Use this to identify a common core of knowledge, if it exists.
- Examine the core knowledge summarized in the *Principles to Guide the Design and Implementation of Doctoral Programs in Mathematics Education* and see how the core knowledge aligns with the syllabus/content descriptions of the graduate level courses for doctoral students in mathematics education.
- How do programs that focus on preparing researchers in mathematics education differ from institutions that focus on preparing collegiate teachers of mathematics? Are there differences in course requirements? Internships? Clinical experiences? Job opportunities?
- Determine how the pathway to a doctorate in mathematics education is different at the same institution when earned in a mathematics department or in a college/school of education. How are the pathways similar/different across several institutions? What are the career aspirations of the respective programs' graduates?
- Identify institutions with doctoral programs in mathematics education that have established intellectual communities and investigate their nature and effect on their program and doctoral students.
- Select a mathematics education doctoral program that has shown significant growth in number of graduates during the last decade and carefully examine the factors that facilitated the growth. Select a program that has declined in number of graduates during the last decade and document the factors that contributed to its decline.
- Survey doctoral graduates in mathematics education to learn about how their doctoral preparation aligned with their post-graduation job expectations.
- Explore accreditation systems in other similar education and non-education areas. Identify arguments for and against an accreditation system for doctoral programs in mathematics education.
- Identify institutions that have invited external examiners to review their doctoral program and document the nature of the review as well as how the review process has impacted their doctoral program in mathematics education.

Conclusion

Our paper has reported some research findings related to doctoral programs and doctoral preparation in mathematics education. We noted an increasing number of institutions producing doctorates in mathematics education. Very little information is known about the nature and quality of the over 125 programs that have graduated doctorates in mathematics education during the last five years. While some programs graduate doctorates in mathematics annually, the overwhelming majority of institutions graduate one student every few years. The small number of annual graduates raises questions about the resources available to provide focused course work related to mathematics education and valuable research experiences. Do institutions graduating someone ever few years have a viable doctoral program in mathematics education? Research is needed to examine and learn more about doctoral preparation programs and the extent to what core-knowledge exists among all doctoral graduates in mathematics education. Accreditation has been suggested as a means of gathering more detailed information from institutions about the nature and scope of their doctoral program in mathematics education.

In an earlier JRME Research Commentary, Schoenfeld focused on the need for and value of replications of research in mathematics education (Schoenfeld, 2018). We agree that replication of research is valuable. However, we argue that so little research related to doctoral preparation in mathematics education has been reported in scholarly journals, that it is a bit early for replication.

Simply put, much more research focusing on multiple aspects of doctoral programs in mathematics education is needed.

Our paper has made clear that limited research on a few facets of doctoral programs in mathematics education has been reported. We offered some possible directions for future research. Our hope is that this paper will stimulate discussion in the mathematics education community that will lead to more research focusing on various components of doctoral preparation in mathematics education. This could include faculty members actively involved in doctoral mathematics education programs doing case studies, i.e., self-examination, of their own or other programs and then sharing their process and what has been learned both internally and externally. This may encourage faculty members in different institutions to collaborate and spearhead efforts that might move toward accreditation of institutions purporting to have doctoral programs in mathematics education. We hope in the future there will be many quality research studies focusing on different aspects of doctoral programs in mathematics education. Such research could provide much needed foundational knowledge to guide the preparation of future generations of stewards of our discipline of mathematics education.

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References

Association of Mathematics Teacher Educators (AMTE). (2003). *Principles to guide the design and implementation of doctoral programs in mathematics education.* San Diego, CA: AMTE.

Bay, J. (2001). Appropriate preparation of doctoral students: Dilemmas from a small program perspective. In R. Reys & J. Kilpatrick (Eds.) *One Field, Many Paths: U.S. Doctoral Programs in Mathematics Education,* (pp. 129–132), Washington, DC: American Mathematical Society/Mathematical Association of America.

Donoghue, E. (2001). Mathematics education in the United States: Origins of the field and the development of early graduate programs. In R. Reys & J. Kilpatrick (Eds.) *One Field, Many Paths: U.S. Doctoral Programs in Mathematics Education,* (pp. 3–18), Washington, DC: American Mathematical Society/Mathematical Association of America.

Dossey, J. & Lappan, G. (2001). The mathematical education of mathematics educators in doctoral programs in mathematics education. In R. Reys & J. Kilpatrick (Eds.) *One Field, Many Paths: U.S. Doctoral Programs in Mathematics Education,* (pp. 67–73), Washington, DC: American Mathematical Society/Mathematical Association of America.

Foley, G. D. (2014). Mentoring doctoral students. In A. Howley & M. B. Trube (Eds.), *Mentoring in the professions: Orienting toward the future* (pp. 205–221). Charlotte, NC: Information Age.

Glasgow, R. (2000). *An investigation of recent graduates of doctoral programs in mathematics education* (Unpublished doctoral dissertation). University of Missouri: Columbia.

Golde, C. (2008). Creating a broader vision of doctoral education: Lesson from the Carnegie initiative on the doctorate. In R. Reys & J. Dossey (Eds.) *U.S. Doctorates in Mathematics Education: Developing Stewards of the Discipline,* (pp. 53–63), Washington, DC: American Mathematical Society/Mathematical Association of America.

Golde, C. & Walker G. (Eds.). (2006). *Envisioning the future of doctoral education: Preparing stewards of the discipline.* San Francisco, CA: Jossey-Bass.

Hiebert, J., Kilpatrick, J., & Lindquist, M. (2001). Improving doctoral programs in mathematics education. In R. Reys & J. Kilpatrick (Eds.) *One Field, Many Paths: U.S. Doctoral Programs in Mathematics Education,* (pp. 153–162), Washington, DC: American Mathematical Society/Mathematical Association of America.

Hiebert, J., Lambdin, D., & Williams, S. (2008). Reflecting on the conference and Looking toward the future. In R. Reys & J. Dossey (Eds.) *U.S. Doctorates in mathematics Education: Developing Stewards of the Discipline*, (pp. 241–254), Washington, DC: American Mathematical Society/Mathematical Association of America.

Inglis, M. & Foster, C. (2018). Five decades of mathematics education research. *Journal for Research in Mathematics Education, 49*(4), 462–500.

Kilpatrick, J. & Spangler, D. (2016). Educating future mathematics education professors. In L. D. English & D. Kirsher (Eds.), *Handbook of International Research in Mathematics Education*, (3rd ed). (pp. 297–309). New York: Routledge.

Lappan, G., Newton, J., & Teuscher, D. (2008). Accreditation of doctoral programs in mathematics education. In R. Reys & J. Dossey (Eds.) *U.S. Doctorates in mathematics Education: Developing Stewards of the Discipline,* (pp. 215–222), Washington, DC: American Mathematical Society/Mathematical Association of America.

Levine, A. (2007). *Educating researchers.* Princeton NJ: The Education Schools Project.

McIntosh, J. & Crosswhite, J. (1973). *A survey of doctoral programs in mathematics education,* ERIC: ED091250. Retrieved from https://eric.ed.gov/?q=ED091250&id=ED091250

Reys, R. E. (2002). Mathematics Education Positions in Higher Education and Their Applicants: A Many to One Correspondence, *Notices of the American Mathematical Society, 49*(2), 202–207.

Reys, R. (2017). Doctorates in mathematics education-How they have evolved, what constitutes a high-quality program, and what might lie ahead? In J. Cai (Ed.). *Compendium for Research in Mathematics Education.* (pp. 934–948). Reston, VA: National Council of Teachers of Mathematics.

Reys, R. (2018a). The Preparation of a Mathematics Educator: The Case of Carey. *Canadian Journal of Science, Mathematics and Technology Education. 18*(1), 56–67.

Reys, R. (2018b). Doctoral programs in mathematics education: Should programs be accredited? *Journal of Mathematics Education at Teachers College. 9*(1), 15–21.

Reys, R. E. & Dossey, J. A. (Eds.). (2008). *U.S. Doctorates in Mathematics Education: Developing Stewards of the Discipline,* Washington, DC: American Mathematical Society/Mathematical Association of America.

Reys, R., Glasgow, R., Ragan, G. & Simms, K. (2001). Doctoral programs in mathematics education in the United States: A status report. In R. Reys & K. Kilpatrick (Eds.) *One Field, Many Paths: U.S. Doctoral Programs in Mathematics Education,* (pp. 19–40). Washington, DC: American Mathematical Society/Mathematical Association of America.

Reys, R. E., Glasgow, R., Teuscher, D., & Nevels, N. (2008). Doctoral programs in mathematics education in the United States: 2007 Status Report. *Notices of the American Mathematical Society, 54*(10), 1283–1293.

Reys, R. E. & Kilpatrick, J. (Eds.). (2001). *One Field, Many Paths: U. S. Doctoral Programs in Mathematics Education,* Washington, DC: American Mathematical Society/Mathematical Association of America.

Reys, B. & Reys, R. (2017). Strengthening Doctoral Programs in Mathematics Education: A Continuous Process. *Notices of the American Mathematical Society, 64*(4), 386–389.

Reys, R. & Reys, B. (2016). A recent history (5 decades) of the production of doctorates in mathematics education, *Notices of the American Mathematical Society, 63*(8), 936–939.

Reys, R., Reys, B., & Estapa, A. (2013). An update on jobs for doctorates in mathematics education at institutions of higher education in the United States, *Notices of the American Mathematical Society, 60*(4), 470–473.

Reys, R., Reys, B., Shih, J. & Safi, F. (2019). Doctoral programs in mathematics education: A status report of size, origin of program leadership, and recommended institutions. *Notices of the American Mathematical Society, 66*(2), 212–217.

Shih, J. Reys, R., & Engledowl, C. (2016) Profile of research preparation of doctorates in mathematics education in the United States. *Far East Journal of Mathematical Education. 16*(2), 135–148.

Shih, J., Reys, R., Reys, B. & Engledowl, C. (2019). Examining career paths of doctorates in mathematics education working in institutions of higher education. *Investigations in Mathematics Learning, 11*(1), 16–28.

Schoenfeld, A. (2018). On replications. *Journal for Research in Mathematics Education, 49*(1), 91–97.

Soonabend, T. (1981). *A survey of doctoral programs in mathematics education,* ERIC: ED201518. Retrieved from https://files.eric.ed.gov/fulltext/ED201518.pdf

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Quantitative Literacy: Alternative Pathway for College Developmental Mathematics Students

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ABSTRACT Low passing rates in developmental mathematics have been a serious concern for community colleges for many years. A course in Quantitative Literacy (QL) offers non-STEM students an alternative option to introductory algebra as a path to a degree. This paper describes the implementation and evolution of QL at the Borough of Manhattan Community College. Students enrolled in the 17 sections of QL were compared to a matched sample of students from Elementary Algebra. The students enrolled in QL in the Spring of 2013 were 175% more likely to have passed a credit-bearing mathematics course one year later, indicating that QL represents a valuable alternative for non-STEM college students placed into algebra level remediation. Further, the implementation and preliminary results of a corequisite course combining QL with college level Quantitative Reasoning (QR) are presented.

KEYWORDS *corequisite mathematics, developmental mathematics, innovative pedagogy, quantitative literacy*

Introduction

Quantitative literacy (QL) has become an increasingly frequent topic of discussion in mathematics education over the last thirty years. Thanks in great part to the efforts of Lynn Steen, author of *Mathematics and Democracy: the Case for Quantitative Literacy* (2001) and *Achieving Quantitative Literacy: an Urgent Challenge for Higher Education* (2004), more and more educators have come to recognize the importance of preparing students for the quantitative challenges they will face in their careers and lives. However, the traditional mathematics curriculum, whether at the secondary or college level, has remained firmly aligned with the longstanding tradition of the calculus trajectory. This is particularly true for students in the standard remedial algebra course at community colleges around the country, many of whom are not bound for STEM-related majors and careers, and whose success rate in this course is egregiously low. According to a study performed by Achieving the Dream with a group of 57 participating community colleges, only a third of students placed into remedial mathematics in these colleges had progressed onto college level mathematics within three years (Bailey, Jeong, & Cho, 2010).

Borough of Manhattan Community College (BMCC) is one of twenty-four institutions comprising the City University of New York (CUNY) and serves over 23,000 students (BMCC Fact Sheet, 2019). Each year about 72% of BMCC'S new entering students are placed into developmental mathematics classes based on their performance on the placement math proficiency test. There are up to three levels of mathematics proficiency that students may need to demonstrate before enrolling in the credit bearing class. Those levels are arithmetic, elementary algebra (EA) and intermediate algebra. A student who is placed into EA or who successfully completes an arithmetic course has a choice to enroll in quantitative literacy (QL) instead of EA if their major does not require more advanced algebra-based mathematics courses (non-STEM students). Figure 1 (next page) illustrates the consistently low success rates of students enrolled in EA.

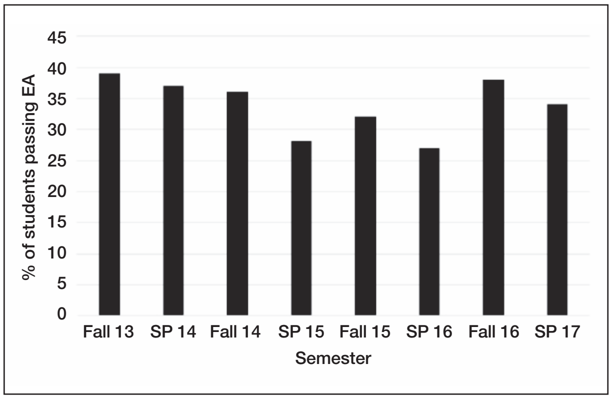

Figure 1. Success Rates in Elementary Algebra Fall 2013 – Spring 2017 semesters.

The QL pathway was introduced in 2012 as a result of BMCC's partnership with the Carnegie Foundation's Quantitative Literacy Initiative together with eight other community colleges across the country. The course was revamped in the Summer of 2015 by a team of BMCC instructors. The curriculum was rewritten to make content more relevant to BMCC students. This course eschews traditional lecture and complex algebraic computation in favor of collaborative work and open-ended problems situated in three applied contexts: citizenship, personal finance, and medical literacy. Problem-solving scenarios include population growth and density, the water footprint of major countries, the cost of an unlimited subway pass vs. single ride passes, interpreting percentages from contingency tables, cost of running a business, representational democracy in the U.S., blood alcohol content, introduction to probability, medical dosage, and compound interest. These problem situations are designed around developing specific mathematical concepts, including estimation strategies, proportional reasoning, understanding magnitude in large numbers, interpretation of probabilities, relative and absolute change, interpreting measures of central tendency, producing and interpreting graphs, calculating quantities using unit analysis, understanding variables, using formulas, solving linear equations, and linear and exponential modeling.

As noted earlier, the course employs an innovative pedagogy designed to support its curriculum. The central motif of this pedagogy is "productive struggle" (Schmidt & Bjork, 1992), wherein students grapple with ideas that are comprehendible but not yet well formed, and which has been shown to lead to greater retention in learning (Hiebert & Grouws, 2007). By facilitating learning, rather than supplying step-by-step algorithms, the QL course facilitates deep learning, develops students' tenacity in problem-solving, and builds quantitative habits of mind, supporting precisely the student-centered approach to learning recommended by Beyond Crossroads (American Association of Two-Year Colleges [AMATYC], 2006).

BMCC's Quantitative Literacy course (QL), which employed the Quantway® curriculum and pedagogical model, required a comprehensive faculty development program which was developed. Between 2012 and 2017 a total of seven faculty trainings were conducted. Over 70 part-time and full-time faculty members participated allowing gradual course expansion.

The QL course has had consistently higher passing rates compared to EA. Table 1 provides the cumulative results in passing rates between the two courses. Figure 2 illustrates semester to semester comparison.

Table 1

Students Performance in Elementary Algebra and Quantitative Literacy courses Fall 2013 – Spring 2017.

Groups	Passed
Elementary Algebra N=17088 (743 sections)	**34%** 3901
Quantitative Literacy N=3052 (122 sections)	**57%** 654

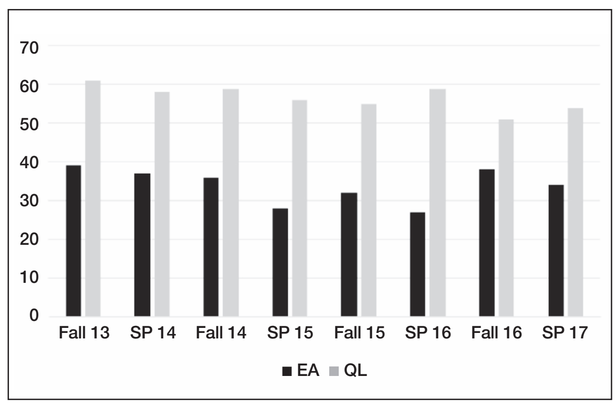

Figure 2. Students Passing Rates in QL and EA between Fall 2013 and Spring 2017 semesters.

The Quantitative Literacy group passing rates have significantly exceeded those of the corresponding developmental algebra course for all semesters the course was offered, with a statistically significant difference in passing rate ($p < 0.001$). A *Chi*-square statistics test showed a highly significant difference between the two groups' passing rates of 34 and 57 percent respectively.

However, these results were not entirely conclusive, as the exit criteria for these courses involve different assessment forms and content. To fully measure success of

QL relative to EA, students must be tracked beyond QL to assess their success in the subsequent college-level mathematics course. In particular, since students taking QL may exhibit different characteristics than the students taking algebra, the QL students' success rate of credit-bearing mathematics completion must be compared to that of a matched sample of elementary algebra students.

Methodology

Subjects & Settings

Students who were in need of remediation at the elementary algebra level and who registered for QL or EA in the Spring 2013 were the target population of the study. Both courses are zero credits and meet two days per week for one hour and 40 minutes each. All basic skills mathematics courses have an enrollment cap of 25 per class at BMCC. All EA sections used the same textbook supplemented by an online homework system. In order to pass EA, students must score a 60 or better on a standardized computer-based final exam and achieve a 74 overall average in the course. At the time of this study, all QL sections used the Quantway® curriculum, supplemented by an online homework system designed for the course and supplied by the Carnegie Foundation. In order to pass the QL course, students must score a 60 or better on the standardized paper final exam and achieve an average of 70 or better in the course.

The Spring 2013 QL cohort of 418 students was deemed the first one large enough to provide statistically persuasive results and an opportunity to assesses the performance beyond the completion of the course.

The students who enrolled in QL in Spring 2013 were distributed across most of the major degree programs offered at BMCC. The largest program majors represented proportionally in the QL group were as follows: Liberal Arts (46%), Criminal Justice (19%), Health/Nursing (14%), and Human Services (10%). Likewise, the EA students were distributed across BMCC's 23 major programs.

Research Design

Students who had enrolled voluntarily in QL differed from the EA students on several key characteristics at the beginning of the Spring 2013 semester:

- The QL group was slightly less likely to be first-time freshmen
- The QL group was more likely to be female
- The QL students had lower arithmetic placement scores (COMPASS 1); COMPASS is a standardized mathematics placement exam
- The QL group had higher average cumulative GPA
- The QL group had higher average total credits accumulated

Since several of these differences could explain higher performance on the part of the Quantitative Literacy group, a propensity matching algorithm was employed to account for potential confounding variables between the two cohorts (Rosenbaum & Rubin, 1983). In lieu of a randomized study, use of propensity matching creates a synthetic balance between the two student groups, and inferences will be valid if there are no residual confounding variables due to unobserved covariates that could substantially bias the results. A sub-sample of 418 propensity-matched algebra students was selected from the overall group of algebra students using *Thoemmes'* propensity matching algorithm for SPSS (Thoemmes, 2012). The algorithm employed the following set of covariates: age, reading score, high school grade point average, first time freshman status, COMPASS 1 score, COMPASS 2 score, gender, underrepresented minority status, cumulative GPA coming into the semester, and total credits coming into the semester. The algebra students were matched to the QL students in equal numbers. Selection was determined by identifying the students that were most closely matched for the above list of covariates. After the matching process, there were no significant differences between the QL and EA sample groups on the demographic and prior performance indices.

At the City University of New York, a student must complete a college level mathematics course with grade C or better in order to transfer to a four-year CUNY campus. Both the QL course group and its EA matched group were assessed on the basis of the completion of the next sequential mathematics course. For students in QL course, the next sequential mathematics course was a credit-bearing mathematics course. For this group, success is defined as achieving a grade of C or better. For students in the EA course, the next sequential mathematics course was either a credit-bearing mathematics course or, in the case of 20 students (4.8% of the sampled population), an Intermediate Algebra course. For these 20 students, success in the next sequential mathematics course was defined as a passing grade in Intermediate Algebra.

Data and Results

Comparisons of the course pass rates were conducted for QL and matched EA students. Table 3 shows the pass rates for the QL and EA groups, both after and before the propensity matching process.

Table 2

Quantitative Literacy and Elementary Algebra, Spring 2013 Pass Rates

	QL (N = 418)	EA Matched (N = 418)	All EA Students (N = 2433)
Passed	**53%**	29%*	33%*

* Fisher's exact test shows these differences between QL and EA pass rates to be significant at a level of p = .000.

Further comparisons were made among students who did not pass the course.

At BMCC students who do not pass a course can be categorized as the following:

- Student stayed in the course the entire semester yet failed to meet the standards for passing the course (F grade).
- Student unofficially withdrew from the class, meaning they stopped coming to class and did not take the final exam (WU grade).
- Student officially withdrew from the class (W grade).
- Student never showed up for class.

Comparisons were conducted in terms of each category for QL and matched EA students. The results are shown in Table 3.

Table 3

Quantitative Literacy and Elementary Algebra, Spring 2013 Pass Rates with Categorization of Unsuccessful Students

	Quantitative Literacy (N = 418)	Elementary Algebra (Matched) (N = 2433)
Pass:	**53%**	**29%**
Fail:	**29%**	**51%**
Withdrew Officially	8%	9%
Withdrew Unofficially	8%	9%
Never attended	2%	2%

* Fisher's exact test shows these differences between QL and EA failed rates to be significant at a level of p = .000.

Those students in each cohort who passed their developmental course in the Spring were followed through the Summer and Fall semesters. In Summer 2013, 48 of the QL students and 62 of the EA students enrolled in classes (not necessarily mathematics). This difference was not considered statistically significant in determining the difference in overall passing rates by the end of Fall 2013 semester. At the end of Fall 2013, students from the original cohorts in QL and EA were assessed in terms of whether they had completed their next level mathematics course. Table 4 shows the QL and matched EA groups' mathematics course enrollment and respective course pass rates by the end of Fall semester 2013.

Table 4

QL and EA Matched Cohort Groups Passing Rates by the end of Fall 2013

	QL Group	EA Matched Group
Enrolled in Next Sequential Math Course	159	87
Passed	110	44
% Passed	69%	51%

Discussion

Students enrolled in QL in the Spring 2013 semester were 2.5 times as likely (110/44) to have completed their next sequential mathematics course one year later, compared to a matched sample of EA students. This result is largely the consequence of the greater passing rate in QL compared to EA. Yet students successfully completing QL were more likely than their algebra counterpart to pass their next level course by 69% to 51%. As noted earlier, twenty of the EA students who passed EA went on to take Intermediate Algebra as their next course. Four of these students passed Intermediate Algebra by the end of 2013. If the sample of successful EA students is restricted to exclude these students, hence including only those who went on to a credit bearing mathematics course as their next enrolled course (i.e, 67 instead of 87 enrolled students, and 40 instead of 44 passing students), the percentage of successful QL students who satisfied their college mathematics requirement by the end of 2013 (69%) is greater than the percentage of suc-cessful EA students who satisfied their college mathematics requirement by the end of 2013 (60%). This result suggests that QL may prepare students at least as well as does EA for a credit-bearing mathematics course. This may be explained, in part, by the fact that little of the content of the credit-bearing mathematics courses offered at the 100-level (Liberal Arts Mathematics, Statistics, Quantitative Reasoning, and Nursing Math) involves the intensive symbol-manipulation characteristic of algebra and many of its specific forms (expressions and equations involving polynomial, rational, or radical expressions). In some cases, the content of the QL course seems more applicable to these credit-bearing courses (dimensional analysis and proportional reasoning in the case of Nursing Math; percentages, two-way tables, and probability in the case of Statistics). Furthermore, it could be argued that the pedagogical methodologies employed in QL, in particular the concept of productive struggle, provide students with a stronger foundation in the kinds of general problem-solving that await them in the subsequent courses.

To explain the significantly higher passing rates in QL, compared to those in EA, both content and pedagogy should be considered. It can be argued that students are more motivated to learn mathematics when they perceive that mathematics to be useful and relevant to their lives. In the case of QL, students are grouped with peers within the class, affording the opportunity to engage in a collaborative problem-solving process enriched by the involvement of multiple perspectives and mutual assistance. Students arguably feel more connected with their peers and their class, strengthening their sense of involvement, which has been argued a crucial aspect of motivation and retention in developmental students (Tinto, 1997; Tinto, Russo, & Kadel, 1994).

Limitations

The study is limited by the accuracy of the selected covariates in determining the propensity matched sample of algebra students. Because students were not randomized, there is still the possibility that the two student groups were not balanced with respect to unobserved factors related to the outcome of interest, even with propensity matching.

Furthermore, as stated earlier, the category of "next sequential course" differed slightly between the respective cohorts. For the entire QL cohort, this category was defined as a credit-bearing, college-level mathematics course, whereas for the EA cohort, this category also included twenty students who continued on in the STEM pathway, which requires a second level of algebra remediation, offered as Intermediate Algebra. Of these twenty students, four passed Intermediate Algebra. Given the relatively small scale of this subgroup of students (4.8% of the sampled population of 418 EA students, and 23.0% of the successful EA students), this may be seen as a small limitation. Note that when the EA sample is restricted to exclude these students, QL students are 2.75 times as likely as EA students to pass their subsequent mathematics course (compared to 2.5 times in the case of the unrestricted sample), and successful QL students are still more likely to pass their subsequent mathematics course than successful EA students. However, in future studies, the respective populations of student might be initially restricted to include only students who are not enrolled in STEM majors before applying the matching algorithm.

Conclusion and Recommendations

The experience of Quantway's implementation at BMCC suggests that Quantitative Literacy offers a promising avenue in the effort to address low rates of successful mathematics remediation in college students. Despite the two distinct pathways that a BMCC student placed into developmental mathematics can choose, a significant number of students do not enroll into the credit bearing mathematics course immediately after successful completion of the developmental pathway. As a result, students delay the completion of their degrees. One recent promising strategy in developmental education that addresses this challenge is a corequisite model (Logue, Watanabe-Rose, & Douglas, 2017). Students are placed into a credit bearing course with just-in-time remedial support. One of the corequisite courses that was developed in Fall 2017 combines the developmental 4-hour, 0-credit QL course with a 3-hour, 3-credit QR course into a new 6-hour, 3-credit course titled, "Quantitative Literacy and Reasoning. The curriculum was written by expanding the available QL materials and introducing college level QR topics such as probability, statistics and financial literacy. The developmental curriculum was carefully embedded so the new curriculum is cohesive, providing students adequate support to be successful. Students are provided with a workbook and access to the online homework platform, both free of charge. Two sections of the course were offered in each of Spring 2018, Fall 2018 and Spring 2019 semesters. A total of 120 students were enrolled in these sections and 59% of them passed the course. The statistics are promising, but further analysis is needed to compare these success rates with students enrolled in a traditional two semester sequence.

Based on the experience and outcome of QL at BMCC, it is recommended that community college mathematics departments consider offering a Quantitative Literacy based course as an alternative to an elementary algebra-based course for non-STEM students. This recommendation is made with awareness of the many obstacles and challenges facing the large-scale implementation of QL. One such consideration involves the uncertainty some students may feel about their future career trajectory, and correspondingly their major. If a student decides at a later point to change to a STEM major, this student would be required to return to the developmental mathematics level and take the EA course.

It is likely that the above rationale may have played some role in preventing, before now, the implementation of an alternative curricular pathway such as Quantway®, and of obstructing such implementation in the future. However, it should also be noted that there are numerous curricular implications of a student's potential to change majors, in a variety of subjects, and more likely than not, such a decision will add to the number of courses that student is required to take. In any case, a one-size-fits-all curriculum does not support the standards advocated by the American Association of Two-Year Colleges (AMATYC). In Beyond Crossroads (2006), AMATYC posits that the central curricular challenge to be addressed is that of designing "curricula that address the needs of as many academic paths and disciplines as possible" (p. 38). The three goals of developmental mathematics curricula and program development, as outlined in Beyond Crossroads (AMATYC, 2006) are given below:

- Develop mathematical knowledge and skills so students can successfully pursue their career goals, consider other career goals, and function as successful citizens.
- Develop students' study skills and workplace skills to enable them to be successful in other courses and in their careers.
- Help students progress through their chosen curriculum as quickly as possible.

The QL curriculum is designed precisely around the objective of helping students to function as successful citizens. Its classroom format, involving productive struggle and collaborative learning, models a workplace environment more closely than does the traditional lecture format. The third objective above argues for a curriculum that will facilitate a more efficient progress through one's *chosen* curriculum (emphasis ours), which, based on the success of QL students in the above study, a QL course would seem to support.

Finally, consider the assertion, made in Beyond Crossroads (AMATYC, 2006), that in the case of developmental mathematics curricula, "faculty need to do more than teach the same mathematics again" (p. 41). One might speculate whether the resistance to an alternative pathway such as Quantway® may have, at its origins, motivations that are not directly pertinent to the academic futures of students. Mathematics is a traditional subject by nature, and many instructors can be expected to be resistant to teaching a course so different in content and pedagogical character than that which they are accustomed to teaching (and which characterized their own mathematics education). Here, as is often the case, change will not be easy, but the effort offers the potential for significant rewards to students, colleges, and faculty.

References

American Mathematical Association of Two-Year Colleges. (2006). Beyond crossroads: Implementing mathematics standards in the first two years of college. Memphis, TN.

Bailey, T., Jeong, D. W., & Cho, S.-W. (2010). Referral, enrollment, and completion in developmental education sequences in community colleges. *Economics of Education Review, 29*(2), 255–270.

BMCC Enrollment Factsheet Spring 2019. (2019, June 11). Retrieved November 12, 2019, from https://www.bmcc.cuny.edu/wp-content/uploads/2019/06/Spring-2019-Factsheet.pdf

Hiebert, J. & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In Lester, F. K. (Ed.), Second Handbook of research on mathematics teaching and learning, (pp. 371–404). Charlotte, NC: National Council of Teachers of Mathematics.

Logue, A. W., Watanabe-Rose, M., & Douglas, D. (2017). Reforming remediation: college students mainstreamed into statistics are more likely to succeed. *Education Next, 17*(2), 78–85.

Rosenbaum, P.R. & Rubin, D.B. (1983). The central role of the propensity score in observational studies for casual effects. *Biometrika 701*(1), 41–55

Schmidt, R. A. & Bjork, R. A. (1992). New conceptualizations of practice: Common principles in three paradigms suggest new concepts for training. *Psychological Science, 3*, 207–217.

Steen, Lynn. (2004) *Achieving Quantitative Literacy: An Urgent Challenge for Higher Education*. Washington, D.C.: Mathematical Association of America.

Steen, Lynn (Ed.). (2001). *Mathematics and Democracy: The Case for Quantitative Literacy*. Princeton, N.J.: National Council on Education and the Disciplines.

Thoemmes, F.J. (2012). Propensity score matching in SPSS. Retrieved from http://sourceforge.net/p/psmspss/home/Home/

Tinto, V. (1997). Colleges as communities: Exploring the educational character of student persistence. *Journal of Higher Education, 68,* 599–623.

Tinto, V., Russo, P., & Kadel, S. (1994). Constructing educational communities: Increasing retention in challenging circumstances. *Community College Journal*, *64*, 26–30.

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Mentoring Future Mathematics Teachers: Lessons Learned from Four Mentoring Partnerships

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ABSTRACT Mentoring is an important aspect of mathematics teacher education, and in particular, pre-service teacher education. Faculty at a large Midwestern university developed and refined a mentoring program designed to help pre-service secondary mathematics teachers, called Scholars, become future leaders in mathematics education. This paper describes how faculty mentors leveraged challenges in the mentoring program's early stages based on their reflections and initial mentee outcomes to create a more effective mentoring program. Recommendations based on research and practice are provided for other university programs interested in mentoring future mathematics teachers.

KEYWORDS *mathematics education, mentoring, Noyce, pre-service teachers, teacher education*

A significant number of teachers, more than one-third, leave the profession within a few years of beginning in the classroom, especially in STEM fields (Shaw & Newton, 2014). Teachers report factors such as feeling underprepared, overwhelmed, and under-supported as reasons for leaving teaching (Kent, Green, & Feldman, 2012). There is a large body of research that supports the need for a strong mentoring component in teacher education programs and its impact on teacher commitment, retention, and student achievement. Marshall, McGee, McLaren, and Veal (2011) highlighted the impact university faculty members and advisors can play in STEM students' success. In particular, factors such as mentoring and helping students "navigate their programs of study to be congruent with their interests, career preferences, and post-secondary commitments" were emphasized (Marshall et al., 2011, p. 22).

Mentoring programs for students have been utilized in a variety of institutions and in a variety of settings (Orland-Barak, 2014). Overall, strong mentoring programs have been proven to positively impact future student learning of the teachers who participate in the program and also significantly reduce the attrition rate of new teachers (Ingersoll & Strong, 2011). The mentoring process can positively influence teacher behaviors and classroom practices of novice teachers (Kuzmic, 1994). Additionally, novice teachers who are mentored are more likely to translate their undergraduate learning of empirical-based instructional practice to their classroom teaching (Darling-Hammond, 2000).

Faculty at a large Midwestern university developed a mentoring program for future mathematics teachers as a component of a Noyce teacher recruitment grant, sponsored by the National Science Foundation. The Phase I Noyce Scholarship program built upon ongoing collaborative efforts between the university and local public schools. The main goal of this program was to strengthen and expand the pipeline for preparing mathematics teachers to better meet the demands of local school districts, particularly in high-need schools.

Initially the grant primarily focused on the development of mathematical content and pedagogical knowledge without a mentoring component. Mathematicians and mathematics educators on the faculty leadership team (FLT) agreed that the addition of mentoring support could be beneficial for program participants. The FLT quickly learned that incorporating mentoring would be a critical, yet complex, facet of the program.

In this paper, a detailed description will be provided that chronicles the development of the mentoring program, including the successes reached and the challenges faced. Effective mentoring programs benefit teacher education programs (Ambrosetti, Knight, & Dekkers, 2014); therefore, the goal of this article is to share the efforts undertaken at our institution to inform and support other mathematicians and mathematics educators who are developing, or want to develop, similar programs to support future teachers.

Setting the Stage

Scholar Selection

At the outset of the grant award, students were recruited to apply for the Noyce scholarship. Students submitted applications indicating their interest in teaching and mathematics, their GPA, and letters of recommendation. Part of the application included a personal essay describing their background, why they were interested in entering the Noyce program, and their future goals as a mathematics teacher in a high-need school. This information served as important data for the mentoring selection process, and again as the process was re-evaluated. Selected students for the scholarship program are referred to as "Noyce Scholars," but for the purpose of this paper will be called "Scholars."

Following the initial round of selection, Scholars and the FLT convened to discuss the role of the mentor and the goals of the mentor-Scholar relationship. The initial meeting included answering the following questions: "What does it typically mean to be a mentor?," "What does it typically mean to be a mentee?," and "Why are such relationships important in preparing to be a mathematics education leader?" The FLT (all of whom are mentors), additional faculty mentors, and Scholars were all provided a book on mentoring new teachers written by the National Council of Teachers of Mathematics (Zimmerman, Guinee, Fulmore, and Murray, 2009). Originally, Scholars were given choice and autonomy to request a faculty mentor. While the FLT were already engaged faculty mentors, other requested faculty were approached to serve as Scholar mentors. Scholars were paired with at least one faculty mentor from the Mathematics and/or Teacher Education departments. These pairings were made either by request of the Scholar or by appointment.

The Role of Mentors

Each mentor and Scholar partnership was encouraged to develop a professional development plan. Mentors and Scholars were encouraged to meet regularly discuss, refine, and report progress on their professional development plans. In the plan, each Scholar committed to demonstrating leadership in mathematics education by the following actions:

- Participating in a multi-day culturally responsive teaching workshop each semester
- Providing leadership within the University Math Club (e.g., serving on a service learning task force, running Math Student Circles)
- Actively participating in extra-curricular mathematics activities at the university
    - Math Club (meetings with other undergraduate mathematics students)
    - Cool Math Talks (talks about engaging mathematics)
    - Math Student Circles (sessions with middle school students doing fun math)
- Planning and completing, with direction from the faculty mentor, a senior project related to mathematics education (e.g., undergraduate research, poster or conference presentation at a regional conference)

Besides this initial meeting and feedback, there was no other structural support given. Minimal processes were in place to train or on-board incoming mentors. Additionally, there were no mechanisms in place to ensure initial expectations of participation or regularly-scheduled mentor-Scholar meetings were upheld. For some of the mentor-Scholar pairs, "regular" meetings were a weekly or multiple times per week event. Yet for others, meetings occurred face-to-face once per semester with intermittent email communication. Initially, the FLT wanted to avoid micromanaging mentors who were essentially volunteering their time to support Scholars. However, the FLT became aware, mainly through Scholar realizations and complaints, that there were large discrepancies between mentor expectations.

After further investigation and individual meetings with the FLT, it became apparent that Scholars were indeed having extremely different experiences depending on the mentoring they were receiving. For example, the FLT found that one Scholar with infrequent communication with his faculty mentor had minimal experiences or progress towards his twelve hours per week Scholar commitment. He had started a few initiatives, including providing mathematics support for a chemistry class, but did not follow through on the commitment or sustain his participation. In contrast, another Scholar spent ten hours per week working with the Emerging Leaders Club for predominately Hispanic students at a local high school. This Scholar helped develop college financial plans, engaged students in innovative problem-solving, and organized guest speakers to talk to the students about college preparation and the university mathematics program. While this Scholar's achievements and dedication was stellar, the FLT believed providing a high-quality program with equitable support and high expectations for all Scholars rather than "islands of excellence" was the true goal and mission of the scholarship program and grant expectations.

Mentor Vignettes on Four Initial Scholars

As the first-year mentor-Scholar partnerships continued, challenges with the mentoring program were more and more apparent in individual faculty mentors' reflections and Scholar achievement. Change was absolutely necessary, so the FLT sought to examine the success and challenges of the first year to make informed decisions about programmatic changes in the future.

The following vignettes represent the perceptions of four mentors working with the four originally funded Scholars. The examination of the mentor reflections and Scholar experiences assisted the FLT in evaluating and addressing how to identify issues and work to find resolutions to the mentoring challenges. Each of the Scholar's mentoring experiences taught the FLT valuable lessons to inform not only future practice, but potentially serve other faculty mentorship programs. In particular, each vignette describes the Scholar's background, mentoring experience, and ultimate retention in the Noyce scholarship program.

Scholar Amy

Amy entered the program as a nontraditional Latina. She was pursuing a teaching degree after being at home with young children and then working as a paraprofessional in the public schools. She had experiences working in a dual language program in the public schools and sought to build upon relationships established through that program.

She was paired with a mentor who had served as a professor in a mathematics class. Her mentor encouraged her to work within the building she had familiarity with and to continue to work with students with whom a previous relationship had been established. The mentor met weekly with her, and facilitated Amy's communications with other Scholars. Through Amy's participation with an emerging leaders' program for Hispanic students in a local high school, Amy not only mentored students, but also engaged other Scholars in the program. The role of the mentor in this case was that of oversight and facilitation. The FLT envisioned mentoring would create opportunities for Scholars to take the lead brainstorming innovative ways to grow as a student and future teacher. This mentor-Scholar relationship demonstrated this student-centered vision. Amy took the lead on a project that was of interest to her, involved other Scholars in the project, met with her mentor regularly for support and guidance, and was consistently involved in the Noyce program in ways that would help her become a better teacher in a high-need school district.

Despite the fact that the leadership team thought of Amy and her mentor as role models for other mentor/mentee pairs, as Amy considered applying for the second year of funding through the Noyce program, issues came to light of which faculty were unaware. The lack of consistency between Scholars' outcomes and participation had caused concerns on both the FLT and some Scholars. In comparison to other Scholars, Amy felt that her mentor had pushed her to do more than the other Scholars already. Entering her second year, Amy became aware of increasing Scholar expectations and decided she did not believe she had adequate time to commit to the program and withdrew.

With better dialogue between the FLT and Amy, it is possible that she could have communicated those concerns earlier on and had positive reinforcements that would have allowed her to participate in the program for the second year. Though Amy still finished her mathematics teacher education program, she did not seek employment as a secondary mathematics teacher. The FLT realized that improved communication and clear guidelines for all mentor/mentee pairs was necessary and may have been influential in retaining Amy in the field.

Scholar Andy

Andy entered the program as an immigrant to the United States of America (USA), only four years earlier. He arrived in the USA as a non-English speaker and was uncertain of his career aspirations or opportunities. Through the guidance of a caring high school teacher, he recognized he was particularly strong in mathematics. After experiencing a positive mentoring relationship in high school, he sought opportunities to be a mentor after entering the University. His initial desire was to be paired with a Noyce mentor with whom he could establish a friendly, personal relationship, similar to what he had been inspired by in his past. Andy was paired with a mathematics professor who was not part of the FLT, but purely volunteered as a faculty mentor. Andy and his mentor met infrequently and by the conclusion of the first year, there was minimal evidence Andy was supported or developed through his mentoring relationship.

Andy was involved in many Noyce events, yet did not take initiative or a leadership role. There were initial attempts to create a tutoring program for chemistry students struggling with math. Problems surfaced in this endeavor and the project was dropped. Andy attended many outreach events and activities, but indicated he was not often aware of the opportunities that existed. Andy's desire to help and be a part of events at the University and within the community led the leadership team to develop better means of communication among Scholars and mentors regarding activities being organized. These efforts would not only serve as a means to inform Scholars of events, but also to serve as a mechanism to showcase leadership endeavors initiated by the Scholars.

Andy did apply for the Noyce program in the second year, but due to other scholarship funding sources he was unable to receive significant funding. He agreed to participate on a part-time basis as a Noyce Scholar. Hence, he did not benefit as much from the second-year programmatic mentoring changes, as his requirements were minimal during the second year. Andy graduated and is currently employed as a secondary mathematics teacher in a high-need school. Since graduation, Andy has not maintained contact with his original mentor, but has attended professional development opportunities offered by the program.

Scholar Katrin

Katrin became a Noyce Scholar as a traditional student. She had graduated at the top of her class from high school and her parents had encouraged her to pursue a degree in engineering. By the end of her freshman year, it was clear that her passion was mathematics, not engineering, and she switched her major. She enjoyed working with others and therefore declared a double major in mathematics and education once accepted into the Noyce program.

Katrin requested her mentor be an education faculty member, who was a previous instructor for one of her classes. The faculty mentor agreed to the request, but mentoring meetings were inconsistent. Mentor support mechanisms, beyond the FLT mentors, were not clearly communicated or established to support this mentor/mentee relationship. Unlike Andy, who was self-motivated to participate in any activities made known to the Scholars, Katrin was unengaged from the Noyce program and rarely attended events and activities. The leadership team accepted much of the blame for this situation, as Katrin did not have access to a significant amount of mentoring. It is unclear what contributions Katrin made to the Noyce program and similarly, the impact of Noyce participation on her growth as a student.

Not surprisingly, Katrin decided not to accept a second year of funding from the Noyce program and was uncertain if she would fulfill the obligation to become a mathematics teacher in a high-need school. After initially struggling early in her career and leaving the field, Katrin did eventually return to teaching and continues to teach in a high-need school.

Scholar Stacy

Stacy, like Amy, applied for the Noyce program as a nontraditional student. She had previously earned a graduate degree in health, physical education, and recreation and had been employed in that field as a swim instructor and manager of a pool. She entered the Noyce program with a strong desire to become a high school mathematics teacher, hoping to have a lasting impact on traditionally underserved children.

She requested the assignment of a mentor from the College of Education, and due to the mentor's busy schedule, she was assigned two mentors. A mentor was assigned from the mathematics department, as well as the requested mentor in teacher education. It became clear that the designation of two mentors presented challenges. It was not obvious which requests and suggestions were being given by which mentor, and communication between the mentors did not occur. As a result, a conflict emerged between the student and one of the mentors. The FLT intervened and eventually assigned a different mentor to the student.

On a positive note, the intervention from the leadership team was successful, and taught the team the power of communication in mentoring. Stacy applied for another semester of Noyce funding and was an active Scholar during that time. She is now fulfilling her Noyce obligation by teaching in a high-need school. She has also become a leader at her school and was even invited to present some of her work at an international education conference.

With all of the original Scholars, the FLT noticed a theme of disconnectedness to both faculty mentors and the cohort of Scholars. Each participated in the same

Noyce program; however, each had drastically different experiences from the mentoring perspective. In any case, Andy, Katrin, and Stacy did continue on to teaching and fulfilling their commitment to teach in high-need schools.

Moving Forward with Reform

From each of these vignettes, a great deal was learned about how to better serve future Noyce Scholars. In particular, several changes in the mentoring process were made as a result of lessons learned from the original cohort of Scholars and their respective mentors. These lessons can be used to strengthen other Noyce programs, but in general, these lessons can be applied when mentoring future teachers.

Research to Inform Practice

At the outset of this Noyce teacher recruitment program, an examination of the literature revealed that the success of the mentoring partnerships could be aided by several factors: a) mentor training, b) the careful selection of participants, and c) the need for ongoing evaluations (Ehrich, Hansford, and Tennent, 2004). Barrera, Braley, & Slate (2010) highlight the importance of well-defined goals in mentoring programs interested in retaining beginning teachers in particular. They described difficulties in their mentoring program as including scheduling conflicts, lack of release time, and no guidelines or preparation provided to mentors and mentees. We found similar findings in our Noyce mentoring program as we aimed to prepare pre-service mathematics teachers for high-need school settings.

In the first year of our Noyce program, mentors and mentees were paired either by Scholar request or FLT placement; however, the features of the mentoring relationships, with regard to intent, purpose, intensity, and duration, were not clearly defined (Crisp, Baker, Griffin, Lunsford, & Pifer, 2017; Jacobi, 1991). Inconsistencies were identified in terms of how the expectations of the mentor-Scholar partnerships were defined. This led to mixed outcomes with the initial group of Scholars with regard to retention and participation. Our experiences parallel other studies on the conceptual limitations of mentoring programs (Crisp, et al., 2017). Through the analysis of mentor reflections after the first year, it was clear that the FLT needed to provide additional structures and supports both for mentors and mentees. The goal was to create more targeted and impactful faculty-Scholar interactions (Museus & Neville, 2012) that met both the professional and personal needs of the Scholars (Jacobi, 1991; Murdock, Stripanovic, & Lucas, 2013) to increase retention of Scholars in the Noyce program.

The main purpose of this paper was to better understand how mentoring can be structured to maximize the potential benefits on undergraduate mathematics education students with regard to promoting academic success and retention. The examination of the individual vignettes revealed problems and successes of the mentoring process, which will be expounded upon in the following section. A critical analysis of past experiences can assist a FLT in being able to learn from issues, resolve them, and create a better program.

Changes in the Mentoring Process

Upon reflection and analysis of the four Scholar experiences and continued mentoring conversations, the FLT identified three main categories of focus for immediate modifications for programmatic improvement and sustainability. These changes were based on research informed best practice of undergraduate mentoring (Zimmerman, Guinee, Fulmore, & Murray, 2009), Scholar feedback, and FLT perceptions based on the needs and goals of the Noyce grant program. Within each category, time and structure were invested to execute a plan for improvement.

Mentoring guidelines established and sustained. There appeared to be a need to establish mentoring guidelines for both the mentors and the Scholars. This would allow for consistency among Scholars as well as provide guidance for new mentors. The first mentoring guideline established focused on mentor-Scholar pairing. While the FLT invited Scholar input on their preferred mentor, they realized that ultimately the successful pairing of supportive mentors should be decided by the leadership team. The FLT must consider Scholar strengths and areas of growth to match each Scholar with a strong, supportive mentor. For example, under new mentoring practices, the FLT intentionally paired an extremely strong mathematics student with minimal teaching experience with a Teacher Education faculty member to provide additional opportunities to work in schools and with children.

Another new mentoring guideline was to create norms for mentoring interactions in terms of frequency and duration. Weekly meetings between each mentor-Scholar pair were to last approximately one hour and provide a forum for open dialogue to occur, addressing not only the ongoing Noyce projects, but also the academic progress and social-emotional well-being of the

Scholar. With weekly meetings, mentors have been more aware and available to support Scholars who navigate busy schedules and take heavy and challenging course loads.

Communication outlets. To sustain the new mentoring guidelines, the FLT now has a variety of outlets to communicate with both mentors and Scholars about the mentoring component expectations. Along with weekly mentor-Scholar meetings, each semester mentors are invited to attend a lunch where faculty update the group on their Scholar projects, share best practices in goal setting and mentorship, and reiterate mentor and Scholar expectations. Additionally, mentors and Scholars are invited to a breakfast at the start of each semester where expectations are shared to ensure all parties hear the common vision and goals of the program.

In addition to regularly scheduled meetings, the FLT created an online communication outlet within the pre-existing campus learning management system to centralize all Noyce communication, announcements, documents and protocols, and activity calendars. All mentors and Scholars have access to upcoming events and expectations in a familiar and frequently utilized campus tool.

Consistency and accountability. Based on the initial four Scholars, it became apparent to the FLT that developing high, consistent expectations for all participants was extremely important. Mentoring was the key link to ensure that Scholars were all striving towards the same target of excellence as they prepared for a career in teaching mathematics. In their reflective journals, scholars collaboratively shared (in an online discussion board) their weekly hours and implications their Noyce activities may have on their future as a teacher and/or mathematician. They list their accomplishments and their challenges each week. Mentors could readily view what the Scholars had written and Scholars became aware of what their peers were doing. The added level of positive peer pressure to the requirements has resulted in a substantial increase in Scholar participation and completion of weekly journaling. For mentors who sometimes struggle to brainstorm collaborative activities with their Scholar, the shared space provides a great resource of project ideas.

In terms of the collaborative mentor-Scholar projects, Scholars are each expected to lead a major project one time per year. These projects are often designed and implemented with their mentors. Being clear on this expectation has led to many important developments including teaching assistantships, undergraduate research projects and articles, community STEM outreach events, conference presentations, and large-scale mathematics events for local high school students. Being purposeful with our mentor-Scholar pairing has also honored faculty mentors who volunteer their time and talents. Both faculty and Scholars can be successful when all parties find mutually beneficial aspects to the relationship.

Conclusions and Future Directions

The lessons learned provoked the establishment of more consistent guidelines, communication, and accountability. In turn, Scholars have shared that collaborating with their mentors and members of the FLT has increased their confidence speaking to "authority/superiors" and also prompted them to take on more leadership responsibilities where they would need to "speak up more and take charge of my ideas." Noyce Scholars share how working with faculty has increased their own perception of their professionalism. With increased leadership expectations reiterated by their mentors and other faculty, our Scholars have been exposed to snippets of diverse experiences intended to strengthen their knowledge and skills as mathematicians and future teachers.

The programmatic changes to mentoring have made a substantial difference in the consistency and implementation of the program as it has continued to grow. Mentoring literature (e.g., Ambrosetti et al., 2014) and critical reflection on our past experiences permitted the FLT to review and revise our practices for increased retention of both Scholars and also our faculty mentors. Of the 12 students who have graduated from our institution and participated in the Noyce program, 11 earned their teaching certification and are fulfilling their commitment to teach mathematics in high-need schools. Developing a deeper connection with faculty and their cohort of peers has resulted in a more collective and collaborative community of learners. Unlike the experiences of Amy, Andy, Katrin, and Stacy, our program, driven by strong mentoring relationships, has become more cohesive and focused.

The Noyce scholarship program continues to develop. Our preliminary research serves to inform future efforts to study the impact of our mentoring program on Scholars over time. The FLT has a deliberate focus on guiding new faculty mentors and members of the Noyce leadership team with clear goals and expectations. As faculty, we continue to recognize how our continued collaboration and willingness to take risks as life-long learners is making an impact on our student Scholars.

We continue to believe and practice that modeling adaptive and reflective behavior will benefit our Scholars as future classroom teachers as they grow dedicated practitioners and leaders themselves. We hope our lessons learned in researching our mentoring program will serve other universities and mentoring programs.

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References

Ambrosetti, A., Knight, B. A., & Dekkers, J. (2014). Maximizing the potential of mentoring: A framework for pre-service teacher education. *Mentoring & Tutoring: Partnership in Learning, 22*(3), 224–239.

Barrera, A., Braley, R. T., & Slate, J. R. (2010). Beginning teacher success: An investigation into the feedback from mentors of formal mentoring programs. *Mentoring & Tutoring: Partnership in Learning, 18*(1), 61–74.

Crisp, G., Baker, V., Griffin, K., Lunsford, L. G., & Pifer, M. (2017). *Mentoring undergraduate students. ASHE Higher Education Report, 43*(1), 1–117.

Darling-Hammond, L. (2000). *Solving the Dilemmas of Teacher Supply, Demand, and Standards: How We Can Ensure a Competent, Caring, and Qualified Teacher for Every Child.* National Commission on Teaching & America's Future, Kutztown Distribution Center, PO Box 326, Kutztown, PA 19530-0326.

Ehrich, L. C., Hansford, B., & Tennent, L. (2004). Formal mentoring programs in education and other professions: A review of the literature. *Educational Administration Quarterly, 40*(4), 518–540.

Ingersoll, R., & Strong, M. (2011). The impact of induction and mentoring programs for beginning teachers: A critical review of the research. *Review of Educational Research, 81*(2), 201–233.

Jacobi, M. (1991). Mentoring and undergraduate academic success: A literature review. *Review of Educational Research, 61*(4), 505–532.

Kent, A. M., Green, A. M., & Feldman, P. (2012). Fostering the success of new teachers: Developing lead teachers in a statewide teacher mentoring program. *Current Issues in Education, 15*(1), 1–17.

Kuzmic, J. (1994). A beginning teacher's search for meaning: Teacher socialization, organizational literacy, and empowerment. *Teaching and Teacher Education, 10*(1), 15–27.

Marshall, S. P., McGee, G. W., McLaren, E., & Veal, C. C. (2011). Discovering and developing diverse STEM talent: Enabling academically talented urban youth to flourish. *Gifted Child Today, 34,* 16–23.

Murdock, J. L., Stripanovic, N., & Lucas, K. (2013). Fostering connections between graduate students and strengthening professional identity through co-mentoring. *British Journal of Guidance and Counseling, 41*(5), 487–503.

Museus, S. D., & Neville, K. M. (2012). Delineating the ways that key institutional agents provide racial minority students with access to social capital in college. *Journal of College Student Development, 53*(3), 436–452.

Orland-Barak, L. (2014). Mediation in mentoring: A synthesis of studies in teaching and teacher education. *Teaching and Teacher Education, 44,* 180–188.

Shaw, J., & Newton, J. (2014). Teacher retention and satisfaction with a servant leader as principal. *Education, 135*(1), 101–106.

Zimmerman, G. Guinee, P. Fulmore, L. and Murray, E. (2009) *Empowering the Mentor of the Pre-Service Mathematics Teacher.* National Council of Teachers of Mathematics: Reston, Virginia.

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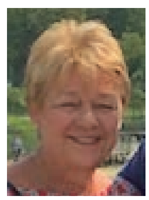

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ACKNOWLEDGEMENT OF REVIEWERS

The Editorial Board would like to acknowledge the following reviewers for their effort and support in reviewing articles for this issue of the *Journal of Mathematics Education at Teachers College.* Without the help of these professionals, it would be impossible to maintain the high standards expected of our peer-reviewed journal.

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This call for papers is an invitation to mathematics education professionals, especially Teachers College students, alumni, and associates, to submit articles describing research, experiments, projects, innovations, or practices in mathematics education. The journal features full reports (approximately 3500 to 4500 words) and short reports (approximately 500 to 1500 words). Full reports describe findings from specific research, experiments, projects, innovations, or practices that contribute to advancing scholarly knowledge in mathematics education. Short reports (previously "Notes from the field") provide examples, commentary, and/or dialogue about practices out in the field of mathematics education or mathematics teacher education; examples from classroom experience are encouraged. Although past issues of *JMETC* focused around a theme, authors are encouraged to submit articles related to any current topic in mathematics education, from which pertinent themes for future issues may be developed. Articles must not have been submitted to or accepted for publication elsewhere. All manuscripts must include an abstract (approximately 150 words in length) and keywords. Manuscripts should be composed in Miscrosoft Word and follow APA format. Guest editors will send submitted articles to the review panel and facilitate the blind peer-review process. Articles for consideration should be submitted online at jmetc.columbia.edu, and are reviewed on a rolling basis; however, to be considered for the fall issue, articles should be received by **February 1st, 2020**.

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