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# JOURNAL OF MATHEMATICS EDUCATION AT TEACHERS COLLEGE

*A Century of Leadership in Mathematics and Its Teaching*

## Reconsidering Elements of Research and Practice

*Published Semiannually by the Program in Mathematics Education*

**TEACHERS COLLEGE** | COLUMBIA UNIVERSITY

## AIMS AND SCOPE

*The Journal of Mathematics Education at Teachers College (JMETC)* is a recreation of an earlier publication by the Program in Mathematics Education at Teachers College, Columbia University. As a peer-reviewed, semiannual journal, it is intended to provide dissemination opportunities for writers of practice-based or research contributions to the general field of mathematics education. Although each of the past issues of *JMETC* focused on a theme, the journal accepts articles related to any current topic in mathematics education, from which pertinent themes for future issues may be developed.

*JMETC* readers are educators from pre-kindergarten through 12th grade teachers, principals, superintendents, professors of education, and other leaders in education. Articles appearing in the *JMETC* include research reports, commentaries on practice, historical analyses, and responses to issues and recommendations of professional interest.

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# JOURNAL OF MATHEMATICS EDUCATION AT TEACHERS COLLEGE

*A Century of Leadership in Mathematics and Its Teaching*

Reconsidering Elements of Research and Practice

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# PROF. BRUCE RAMON VOGELI

JMETC Readers,

This issue of the journal begins on a somber note, as we reflect on the passing of Prof. Bruce Ramon Vogeli earlier this year. Given his immense influence on the Program in Mathematics at Teachers College, Columbia University, and on the revival of the Program's journal (JMETC), the editorial board decided it fitting to write a tribute in his memory.

Since the Teachers College community learned of Prof. Vogeli's passing in May 2020, there have been myriad tributes to him highlighting his contributions to the field of mathematics education and his half-century of service to Teachers College. In addition to these accolades, the editorial board felt it important to highlight his enormous dedication to, and impact on, students in the Program in Mathematics.

Prof. Vogeli was tirelessly devoted to the Program and its students—inevitably, he was the first to arrive, and the last to leave the office. He expended endless energy devising creative ways to ensure the program and all of its projects ran smoothly and effectively; he always had a new idea or scheme to attain the absolute best from students. And he was never too busy for even the most trivial of questions. Even a 30-second question might turn into a 30-minute conversation. He cared deeply for students' wellbeing, and relied on his decades of experience in the field to find creative solutions to difficult problems in students' academic, and even personal, lives. He incessantly encouraged students to achieve their very best and, in this, he demonstrated full confidence in their abilities—even for tasks students themselves thought impossible. His vocal support inevitably resulted in an assurance and trust in their own capabilities along the way.

Prof. Vogeli dedicated an immense amount of time to creating opportunities and experiences for students—ones that might advance their, not necessarily his, professional careers. He had a particularly special ability to incorporate students into his own projects, and to tap into individual students' interests and help them turn these into fruitful ideas on which to base papers and curricular materials. Consistently, he created projects that involved students in his own work, often giving the students ownership of projects he helped envision. Several of his edited volumes feature chapters that he invited his students to write. He also devised and led the effort to create handbooks of curricular materials, handing off the writing and editing to students and alumni, maintaining only an advisory role. Indeed, this journal itself, the Journal of Mathematics Education at Teachers College, was revitalized by Prof. Vogeli from an earlier departmental publication, specifically with the goal of providing students and alumni from the program more opportunities to research, write, and publish their work. He fostered leadership skills through this journal, by appointing students to serve as guest editors and allowing them to oversee the entire peer-review publication process. It is not hyperbole to say that this journal, from its inception, was centered around advancing the professional capacities, opportunities, and experiences for students in the Program. Prof. Vogeli's effort and dedication to students in these regards deserves equal praise as to that of his other works in the field; his legacy will continue to impact the field of mathematics education and its future generation of professionals and scholars for many years to come.

The Program in Mathematics, and its community, all owe a great debt of gratitude to Prof. Vogeli for the enormous opportunities he created and the scholars he nurtured.

JMETC Editorial Board
*Nick Wasserman (Chair), J. Philip Smith, Nicole Fletcher, and Hudson Gould*

## On Professor Bruce Ramon Vogeli

*In addition to this short tribute from the editorial board, we also include a survey of Prof. Vogeli's professional life and career, written by his Program colleagues, Profs. Alexander Karp and Erica Walker. This piece, entitled "On Professor Bruce Ramon Vogeli" (pp. 1-5), opened a published volume that honored Prof. Vogeli.* The following is a reprint from that volume. (It is important to note that since the publication of that volume, three more books edited by Prof. Vogeli were published: Special Secondary Schools for the Mathematically Talented, Mathematics and its teaching in the Asia-Pacific Region (with John Mack), and Mathematics and its teaching in the Muslim World (with Mohammed El Tom).)*

**Professor Bruce Ramon Vogeli** has been working at Teachers College, Columbia University since 1964. Over the last half-century, he has taught dozens of courses, written dozens of papers and books, and graduated not hundreds, but thousands of students. Indeed, just the number of doctoral students who defended their dissertations with him as advisor, has long passed one hundred.

Vogeli belongs to the generation born right after those who fought in the Second World War, but the experience of the war may nonetheless have been the most important factor in his development—direct opposition to evil, demanding courage, hard work, self-sacrifice, and education, was crucial for his formation as a human being. Moreover, the peace that came in 1945 was fragile, coming under threat frequently and everywhere in the world, from Berlin to Korea. It is therefore not surprising that Bruce Vogeli began his adult life by serving in the army, with which he was involved—counting his active service and reserve service—for nine years altogether, starting in 1948. Among other things, he served as a technical analyst, obtained his Bachelor's and Master's Degrees, and worked as a schoolteacher of mathematics during this time. In 1957, he enrolled in the graduate program at the University of Michigan (Ann Arbor). Here, his knowledge of the Russian language, which he had learned while in the army, proved useful: his dissertation, which he defended three years later, was devoted to Soviet mathematics education.

The young Doctor of Philosophy began his academic career as an assistant professor at Bowling Green State University, Ohio, becoming an associate professor in a few years. But now, the Russian language once again turned out to be useful. After Khrushchev's visit to the United States in 1959, a decision was made to organize a professor exchange—a Soviet scholar would come and teach a course in the United States, and an American professor would do the same in the Soviet Union. Bruce Vogeli became this American professor, traveling to Moscow in 1963.

It was exactly this period that witnessed the appearance of Soviet schools with an advanced course of study in mathematics, one of the most remarkable phenomena in the global practice of mathematics education. Vogeli observed their appearance and was the first to write about them for educators living outside the Soviet Union. His book, Soviet Schools for the Mathematically Talented, became an important milestone in international mathematics gifted education. It inaugurated a series of publications by Vogeli which addressed Russia, advanced study of mathematics, comparative education, and even more broadly, the development of curricula.

* Karp, A., & Walker, E. N. (Eds.) (2015). *In honor of professor Bruce Ramon Vogeli: Scholarship and leadership in mathematics education.* Bedford, MA: Consortium for Mathematics and its Applications (COMAP).

**On Professor Bruce Ramon Vogeli, *continued***

These topics have continued to occupy Vogeli for much of his life, and perhaps no less importantly, many of his students started working on them as well. Shortly after returning to the United States, Vogeli became a professor at Teachers College, Columbia University, and naturally, his doctoral students became interested in subjects that interested their teacher—one can name dissertations written under his direction on mathematics schools in Hungary and Russia (Identification and Development of the Mathematically Talented: The Hungarian Experience; Schools for the mathematically talented in the former Soviet Union), or dissertations on education in Latin America (Relationships between mathematical education and economic production in six Latin American countries from 1960 to 1970), or dissertations on mathematics education in Africa (Computers in Africa: Survey of availability of computers, trained manpower, and computer education), or dissertations on how to develop and evaluate a curriculum (Transformational geometry in the junior high school: An evaluation of curricular units in 7th Grade; Game theory for the secondary mathematics curriculum)—the list could go on and on. One can say that a school of comparative studies in mathematics education formed at Teachers College, Columbia University—the "Vogeli School". And in this respect, Vogeli became a worthy successor in the tradition initiated by David Eugene Smith, who founded the Program in Mathematics at Teachers College and established the field of comparative studies in mathematics education.

Vogeli also continued the other work of David Eugene Smith and subsequent generations of Teachers College professors. The writing of textbooks is sometimes considered a less meaningful pursuit for a scholar than the writing of research papers. It should not be forgotten, however, that it is precisely textbooks that are read by millions of students and used by thousands of teachers. Without minimizing the role or significance of research papers, we would argue that the experience of research studies and research findings, multiplied by pedagogical experience, and embodied in the form of a generally accessible school textbook, exerts an influence on the surrounding world faster, and often also more effectively than even the most striking scholarly articles, which are read only by a very few. Bruce Vogeli has authored and coauthored many dozens of textbooks—for elementary, middle, and high schools—which have been used by literally millions of children (and not just in the United States, since these textbooks have been translated into other languages.)

Lastly, and understandably, in over fifty years of working at Teachers College, Columbia University, Vogeli has done a great deal both for Teachers College, and for many other universities. Consultations, meetings, guest lectures, Fulbright professorships, and so on, and so forth—he has been invited all over the world to help with the development of mathematics education.

Still, the most important part of his work, of course, has been at Teachers College itself. Here, he has been a member of numerous committees and work groups, and he has chaired departments. But his most important contribution has been as head of the Program in Mathematics. Having at one time become the top place in the country for doctoral degrees in mathematics education, the Program has retained its leading position in the field—more doctoral dissertations are defended here than anywhere else.

There is yet another fact that must be noted. Over the last few decades, people's understanding of what is important for a specialist in mathematics education has changed time and time again. At certain stages,

**On Professor Bruce Ramon Vogeli, *continued***

the importance of a deep knowledge of mathematics was effectively rejected: people simplistically argued that in the future graduates would unlikely need to teach any particularly recondite subjects, and hence had no need to understand them themselves. The fact that a teacher must nonetheless have a sound grasp of a subject in order to teach even its relatively basic sections somehow eluded people's understanding. The result is that now, when no one seems to doubt the importance of content knowledge, including pedagogical content knowledge, many institutions have neither the structure for instruction in the corresponding areas, nor people who are capable of carrying out such instruction, combining mathematical and pedagogical preparedness. The Program in Mathematics at Teachers College, Columbia University has preserved its character, and the credit for this enormous achievement belongs to Bruce Vogeli, who was able both to preserve and to expand important courses, and to find, invite, and educate collaborators.

The fact that Vogeli has had such a long career does not mean that there has been any diminution in his activity. In the last few years alone, he has published a number of books including Russian Mathematics Education: Programs and Practices (coedited with Alexander P. Karp) and Mathematics and its Teaching in the Southern Americas (coedited with Hector Rosario and Patrick Scott). These books both continue the research that Vogeli has carried out earlier, and mark out new paths for study. Bruce Vogeli continues to conduct seminars and teach courses for students, as he has done for decades, and to serve as their dissertation advisor.

# PREFACE

## A Brief Introduction for Posterity

We begin this issue of the *Journal of Mathematics Education at Teachers College (JMETC)* by noting its publication at a most unusual time in the world. The COVID-19 viral pandemic has upended human life in almost every part of the planet. Scholars agree that the influence of this epoch will be seen for decades, if not centuries, to come. At the time of this preface's writing, some of the most essential functions of societies have faced tremendous difficulty and even failure. Therefore, all who have served others during this emergency deserve tremendous praise and gratitude. Among these individuals are teachers, who have been charged with meaningfully teaching the world's future generations during a global crisis. On behalf of everyone at *JMETC,* we wish to sincerely thank—with the utmost sincerity and conviction—educators everywhere for risking their lives and responding valiantly, despite the tremendous associated burden and sacrifice. In honoring and commending the world's teachers, we proceed with the publication of this issue under some of the most challenging and unprecedented circumstances in modern history, so that it may serve as a symbol of educators' unending commitment and resilience. We are therefore humbled to present this artifact to stand in honor of educators across the world.

## Reconsidering Elements of Research and Practice: Some Perspectives

The Fall 2020 issue of *JMETC* presents six research and practice-based articles that invite readers to reconsider research in practice from several different perspectives. Some pieces reconsider traditional instructional materials, lesson design, classroom discourse, and assessment. We also present a paper that reconsiders research in understudied subfields of mathematics education. Together, these contributions provide readers with unique perspectives on contemporary research and practice and offer clear directions for future work in these areas.

To begin this issue of *JMETC,* Hung-Hsi Wu, a leader in the field of mathematics education, challenges our conception of K-12 mathematics textbooks and their role in curriculum and instruction. Through his illustration of how school textbooks present the division algorithm in a manner that promotes mathematical misconception and misunderstanding, Wu argues that K-12 mathematics teaching has been disserved by the textbooks that are used. Accordingly, he calls for a reconsideration of how textbooks are authored, valued, and used in school mathematics. Within his critique, Wu offers possible remedies for the issues he claims are inherent in what he refers to as "Textbook School Mathematics."

# PREFACE (Continued)

Next, Campbell and Yeo explore the nature of the contributions made by students to mathematical discourse in the classroom. Complimenting research focusing on teachers' roles in scaffolding mathematical discourse, Campbell and Yeo offer a theoretical framework for students' responsibilities in these interactions. The authors then utilize this framework to analyze authentic vignettes of student discourse in middle and postsecondary classrooms. Campbell and Yeo's work encourages practitioners at all educational levels to consider students' roles in generating and maintaining meaningful mathematical conversations to promote understanding.

Barba continues the focus on discourse through an investigation of mathematical discourse outside of the classroom. Her analysis contributes to the understudied area of mathematical discourse on social media. Barba reports that social media discourse can reveal meaningful information about individuals' mathematical mindsets and identities. Through a unique investigation of discourse surrounding a mathematics problem posted to a social media site, Barba provides a thought-provoking exploration of what can be learned by individuals' unfiltered interactions online. By framing her findings within the context of existing research, Barba highlights the relevance of her work and describes avenues for future research in this emerging field.

MacMahon and Mongroo continue the issue with a timely analysis of literature on alternative mathematical assessments for postsecondary educators. In detailing online, oral, and project-based assessments, McMahon and Mongroo highlight the potential benefits, drawbacks, and important considerations for using each type of assessment in modern postsecondary classrooms. In this guide for practicing teachers, the authors provide clear and actionable information about non-traditional assessment as well as an easy-to-use checklist designed to help teachers leverage these to assess their students' learning.

Next, we feature our "Notes from the Field" section, which offers short papers detailing classroom practice. In the first piece, Simon describes how her students utilized a dynamic geometry software program to create personalized logos. This classroom episode provides readers with an example of how one practitioner reconceptualized mathematical modeling through the lens of design to engage students in the learning of geometric transformations. In addition to providing lesson details, excerpts from video-recorded lessons, and examples of student work, Simon demonstrates the success of the current task and argues for incorporating more lessons of this type into K-12 mathematics classrooms.

# PREFACE (Continued)

The issue ends with a classroom episode from Wheeler and her colleagues in which preservice and inservice teachers engage in a lesson incorporating innovative robotic technology. Wheeler et al. describe how they engaged students in a graduate mathematics education course in tasks that featured the Sphero BOLT, a small robotic device that can be moved utilizing mobile applications and basic computer code. The authors detail how they demonstrated the teaching of the concepts of velocity, time, and distance within an interdisciplinary lesson context combining supplemental learning in technology, computer science, and even children's literature. Wheeler and her colleagues note how such a lesson helps preservice and inservice teachers develop the capacity to design, structure, and implement such an experience for their future or current students. The authors also provide information for teacher educators interested in incorporating technology such as the BOLT into mathematics education classes at the postsecondary level.

In closing, we note that one of the foundational principles of mathematics education—and education in general—is that improvement stems from reflection, reconsideration, and thoughtful action. The articles presented herewith align with this purpose.

Brian Darrow, Jr.
Anisha Clarke

*Guest Editors*

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# Misconceptions About the Long Division Algorithm in School Mathematics

Hung-Hsi Wu
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**ABSTRACT** The non-learning of school mathematics is now almost universally taken for granted, but this does not have to happen. This article takes a critical look at the root of this non-learning by pointing to the flagrant defects in the kind of mathematics—to be called TSM—that is predominant in almost all the school textbooks. By analyzing how the long division algorithm is taught and why it becomes so mystifying to school students, we explore how to provide the missing reasoning that makes sense of the algorithm. The article also suggests some concrete steps we can take to eradicate TSM.

**KEYWORDS** *algorithm, division with remainder, equal sign, long division, proof, reasoning, Textbook School Mathematics, TSM.*

For roughly the last five decades, our nation has had a *de facto* national school mathematics curriculum, one that has been defined—with *perhaps* a very small number of exceptions in the last five years—by the standard school mathematics textbooks. It is a fact, though not one that has been explicitly discussed in the world of mathematics education, that the mathematics embedded in these textbooks is extremely flawed, to the point of being unlearnable (in the sense of *unlearnable by a majority of students*). It is notable for its lack of definitions for concepts (e.g., what is a fraction and what does it mean to multiply two fractions?), lack of reasoning for skills (e.g., why is negative times negative positive?), almost universal lack of precision (e.g., is "$3^0 = 1$" a definition or a theorem?), a general lack of coherence (e.g., are finite decimals and fractions different kinds of numbers?), and a pervasive lack of mathematical purpose in its presentation (e.g., telling students to learn to take the absolute value of a number by killing the negative sign because this skill will be on the test). For ease of exposition, we call this particular version of school mathematics **TSM** (Textbook School Mathematics). We will refer to pp. 22-30 (of the pagination of the PDF) of Wu, 2020d for a more detailed discussion of TSM. In retrospect, much of the turmoil in school mathematics education during the last thirty years has revolved around disagreements on how best to deal with the absurd situations that arise in our school mathematics classrooms when teachers try to teach something as nonsensical as TSM (Wu, 2020d). For all these reasons and more, it is no longer possible in 2020 to discuss mathematics learning in schools without directly confronting TSM. The central issue now is how to eradicate TSM and help teachers and students, schools, and districts transition to a different version of K-12 mathematics that is transparent, and therefore learnable.

There is no better illustration of the fiasco that is TSM than the multiple defects in how the concept of division is taught in elementary school. The ubiquity of the limerick, "Ours is not to reason why, just invert and multiply," points to the catastrophic failure in the teaching of the *division of fractions*, but less well-known is the fact that a failure of comparable magnitude has already occurred in the teaching of division among the whole numbers. On the one hand, there is the concept of *the division of one whole number by another*, such as $35 \div 7$ or $36 \div 6$, and on the other, there is the concept of the *division-with-remainder of one whole number by another*, e.g., the division-with remainder of 35 by 6 for which the symbol $35 \div 6$ cannot be used. These are two different concepts but

TSM makes believe that they can be conflated. In addition, TSM does not make explicit the fact that the long division algorithm, e.g., of 35 by 6, is a shortcut that yields the division-with-remainder of 35 by 6. These flaws of TSM destroy the bridge that leads from the long division algorithm to at least two topics in middle and high school: the conversion of a fraction to a decimal by "the long division of the numerator by the denominator" (in a sense to be made precise later) and the division algorithms for polynomials. This is but one example of how TSM suppresses the *coherence* of school mathematics and, instead, presents mathematics to students as a collection of fragmented pieces of factoids to be memorized by brute force.

The mishandling of the long division algorithm by TSM in elementary school and the ripple effects of this particular failure in the school mathematics curriculum are the main concerns of this article. It will also offer some suggestions on how to improve the teaching of this algorithm in grades 4-6. In the last section, we put this discussion of the long division algorithm in the broader context of how TSM has made school mathematics a horror story. We will also describe a recent development—the publication of a detailed curricular road map for making K-12 mathematics mathematical—that may eventually render TSM a relic of the past.

## Teaching Long Division in Grade 4

Consider teaching the long division 78 by 4 in grade 4. The usual setup for long division is to draw a "division house" (in the terminology of Green, 2014), putting 78 inside and 4 outside.

$$\begin{array}{r} 19 \\ 4\,\overline{)\,78} \\ \underline{4\phantom{8}} \\ 38 \\ \underline{36} \\ 2 \end{array} \qquad (1)$$

Students are told that the number 19 on the roof and the 2 at the bottom are the answer to the following question: if they want to put 78 apples in groups of 4, how many such groups are there, and how many apples (if any) are left over? They are also taught to write this as $78 \div 4 = 19\ R2$.

Fourth graders undoubtedly have a hard time understanding why the "division house" in (1) gives the correct answer of 19 equal groups of 4 with remainder 2. The effect of the putative "equality" $78 \div 4 = 19\ R2$ on their mathematics learning is, however, more insidious and more lasting. First of all, mathematics education certainly should not engage in teaching something that is blatantly false, but $78 \div 4 = 19\ R2$ is blatantly false. To see this, if we divide 59 by 3, we also get 19 with the remainder 2. So we have $59 \div 3 = 19\ R2$. It follows that $59 \div 3$ and $78 \div 4$ (whatever they are!) must be equal since they are both equal to 19 $R2$. Even fourth graders can sense that, whatever "equality" means, the equality $59 \div 3 = 78 \div 4$ looks really bad. To understand why, we must put ourselves in the context of fractions to see that this implies $\frac{59}{3} = \frac{78}{4}$, which implies $19\frac{2}{3} = 19\frac{2}{4}$, which in turn implies $\frac{2}{3} = \frac{2}{4}$. The last equality is clearly false.

Let us look at $78 \div 4 = 19\ R2$ from a different angle. It would appear that TSM uses it as a shorthand for "do the long division of 78 by 4 and the answer is 19 with remainder 2." It turns out that this kind of illegitimate shorthand is part of a common pattern in TSM. Consider the teaching of fractions in TSM, for example. The equal sign in any of the formulas for arithmetic operation is always used as a call to do a computation or announce the result of a computation (in the following, $a$, $b$, etc., are whole numbers which may be assumed to be nonzero where necessary):

$$\frac{a}{b} \pm \frac{c}{d} = \frac{pa \pm qc}{m} \quad \text{where } m \text{ is the least common multiple of } b \text{ and } d \text{ and } m = pb = qd \qquad (2)$$

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \qquad (3)$$

$$\frac{a/b}{c/d} = \frac{a}{b} \times \frac{d}{c} \qquad (4)$$

None of these formulas are intended by TSM to convey the message that the quantities on the two sides are "equal." Indeed, a correct mathematical exposition would first *define* what it means to add, subtract, multiply, and divide two fractions before *proving* that the fractions on both sides of each of (2) – (4) are the same fraction, i.e., the same point on the number line. (See Wu, 2011, Sections 14.1, 16.1, 17.1, and 18.2.) However, TSM, as a rule, does not provide *definitions* for concepts—it may provide pictures and metaphors, but not mathematical definitions—so that students are left in the dark about what a fraction is and, therefore, also what it means to add, subtract, multiply, and divide two fractions. In TSM, each of these four equations, like $78 \div 4 = 19\ R2$, is nothing more than a command to perform a computation, e.g., (4) says to divide $\frac{a}{b}$ by $\frac{c}{d}$, simply invert $\frac{c}{d}$ and multiply by $\frac{a}{b}$. In TSM, it is irrelevant what "division" means; all that matters is that students get the right answer when called upon to do a division.

To further reinforce our claim that TSM consistently misinforms students about the equal sign, let us look at what TSM says about equations and how to solve them. According to TSM, an equation in one variable is an equality of two expressions involving a "variable" $x$, such as $3x + 1 = x - 5$. The instruction from TSM on how to solve such an equation is to go through the following steps of symbolic manipulations:

**Step A:** $(-x) + 3x + 1 = (-x) + x - 5$.

**Step B:** $2x + 1 = -5$.

**Step C:** $2x + 1 + (-1) = -5 + (-1)$

**Step D:** $2x = -63$

**Step E:** $x = -3$

The answer of −3 is indeed correct, but what do steps A–D *mean*? Take Step A, for example. TSM says it follows from the *equality* $3x + 1 = x - 5$ by adding the same expression $(-x)$ to both sides. But in what sense is $3x + 1 = x - 5$ an equality? Since $x$ is a variable, it can take on arbitrary values such as $x = 1$. In that case, the left side is 4 and the right side is –4, and they are certainly not equal! The same comment applies to Steps B, C, and D. The use of the equal sign in this standard process of solving the linear equation is therefore a *mathematical* travesty. So, once again, what TSM wants is not for students to learn how to use the equal sign correctly but only to know that they should go on automatic pilot to do computations at the sight of the equal sign (For a correct definition of what an equation is and how to correctly solve the equation $3x + 1 = x - 5$ via Steps A–D, see Sections 2.1 and 3.1 of Wu, 2016b.). Wu (2016b) shows how to solve the equation $3x + 1 = x - 5$ correctly via steps A-D in Sections 2.1 and 3.1.

Thus, there should be no mystery about why students fail to understand the meaning of the equal sign: *TSM has systematically corrupted their conception from the outset. Garbage in, garbage out.* This failure has drawn the attention of educators in the past four decades because it has hampered students' ability to learn algebra in middle school (e.g., Falkner et al., 1999, Kieran, 1981, & Knuth et al., 2008). However, the connection between TSM and students' failures in mathematics, particularly algebra, seems to have been overlooked thus far. As mentioned above, one cannot look past TSM in the year 2020 in any attempt to improve student learning, so we hope education research will, at last, recognize the need to eradicate TSM from school mathematics education.

Let us now revisit (1). A key point is how to introduce the division symbol "÷" correctly to students *in the context of whole numbers.* We define $35 \div 5$ to be the whole number $k$ so that $35 = k \times 5$ ( in the same way that we introduce the subtraction $17 - 9$ to be the whole number $m$ so that $9 + m = 17$). Then it is clear that the equation $35 \div 5 = 7$ is correct since $35 = 7 \times 5$. In general, if we know ahead of time that $m$ is a multiple of $n$ ($n \neq 0$), then $m \div n$ is by definition the whole number $k$ so that $m = k \times n$. If, however, $m$ is not multiple of $n$, we are at a loss as to what $m \div n$ could mean as a whole number or two whole numbers. There is, therefore, no way that something like $78 \div 4 = 19\ R2$ could make any sense as an equality about whole numbers—unless you *insist*, as TSM does, that the *computation with the "division house" must have an answer* and "$78 \div 4 = 19\ R2$" is the symbolic expression of choice. School mathematics must reject such bizarre impulses and teach students to rigorously observe that—*in the context of whole numbers*—the division symbol $m \div n$ ($n \neq 0$) can be used only when $m$ is known to be a multiple of $n$. This kind of precision is by no means inappropriate for fourth graders. After all, even second graders learn not to write $5 - 9$, or in general $k - l$, when $k < l$ in the context of whole numbers.

If we do not know whether or not $m$ is a multiple of $n$, then we have to introduce the concept of division-with-remainder. Here is the definition of the **division-with remainder of $m$ by $n$** ($n \neq 0$): it is an expression of $m$ in terms of $n$ and two whole numbers $q$ and $r$ so that

$$m = (q \times n) + r \quad \text{where } 0 \leq r < n \tag{5}$$

The number $q$ is called the **quotient** of the division-with-remainder and $r$ its **remainder** (both the quotient and the remainder are unique, cf. Wu, 2011, pp. 104-105). If the remainder $r$ in the division-with-remainder is 0, then $m$ is a multiple of $n$ and the two concepts of *division of $m$ by $n$* and *division-with-remainder of $m$ by $n$* coincide. We note that the restriction of $0 \leq r < n$ on the remainder $r$ is an essential part of the definition because it guarantees that the whole number $q$ is the largest whole number so that $q \times n \leq m$, as we now explain.

In fourth grade, of course, we define division-with-remainder only by using explicit examples. For the case at hand: the division-with-remainder of 78 by 4 is expressed as

$$78 = (19 \times 4) + 2 \quad \text{where "2" satisfies } 0 \leq 2 < 4 \tag{6}$$

This equation implies that if there are 78 apples (i.e., the left side of (6)), then it is the same number of apples as in 19 groups of 4 apples (i.e., $(19 \times 4)$) plus 2 extra apples on the side (i.e., the +2 on the right side of (6)). *This is the intuitive meaning of "division-with-remainder of* 78 *by* 4" *that we want to convey to students.* If we can teach students how the "division house" in (1) leads directly to (6), then the "division house" will become learnable mathematics rather than just a senseless ritual.

Naturally, there are other expressions for 78 that superficially resemble (6). For example,

$$78 = (18 \times 4) + 6$$

But this is not the division-with-remainder of 78 by 4 because the "remainder" here, 6, does not satisfy the requirement of being less than < 4 as stipulated in (6). So, we take out another "group of 4 apples" among the 6 leftover so that the 18 equal groups of 4 become 19 equal groups, and there are now 2 leftover as in (6). On the other hand, we cannot get 20 equal groups of 4 out of 78 because 20 × 4 = 80, which is greater than 78. Therefore the 19 in (6) is the largest whole number so that (19 × 4) ≤ 78. We usually express this by saying that **19 is the largest multiple of 4 that is ≤ 78.**

By tradition, we continue to call $m$ the dividend and $n$ the divisor in (5). Thus, 78 is the dividend, 4 is the divisor, 19 is the quotient, and 2 is the remainder in the division-with-remainder (6).

Knowing that the quotient is just the largest multiple of the divisor not exceeding the dividend tells us that no thinking is needed to get the division-with-remainder of one number by another. For example, to find the division-with-remainder of 78 by 4, we could simply write out the multiples of 4 until we get close to 78:

$$0, 4, 8, 12, \ldots, 68, 72, 76, 80, \ldots.$$

By inspection, 76 is *that* multiple. So, since 76 = 19 × 4 and 78 – 76 = 2, we see that the division-with-remainder of 78 by 4 is given by (6), i.e.,

$$78 = (19 \times 4) + 2$$

While this way of getting the quotient and remainder may be straightforward, it can get very tiresome very fast: think about getting the division-with-remainder of 78765 by 4 by listing all the multiples of 4 up to and just beyond 78765. We need a shortcut, and the "division house," i.e., the long division algorithm in (1) is that shortcut, as we now show. After all, (1) is a bit more pleasant than listing the multiples of 4 up to 80.

The main purpose of this article is to tell the full story about *why* the long division algorithm in (1) leads inexorably to (6), but grade four may not be the right place to do this. Nevertheless, if we believe in teaching students *mathematics* rather than just procedures, we have to find ways of offering some grade-appropriate reasoning to make sense of (1) to fourth graders (in the language of Wu (2006), we are performing mathematical engineering to make (1) consumable by fourth graders). Section 7.4 of Wu (2011) makes two such well-known suggestions, and we will recap one of them here. For this purpose, having a correct definition of division-with-remainder as in (5) becomes an indispensable asset. First, let us rewrite (1) by putting in the zero that was intentionally omitted for simplicity:

$$\begin{array}{rr} & 19 \\ 4\,) & \overline{78} \\ & \underline{40} \\ & 38 \\ & \underline{36} \\ & 2 \end{array} \qquad (7)$$

This rewrite makes it obvious that the subtraction 78 – 40 = 38 is actually an intermediate step in the long division (the omission of 0's in (7) is of course a common practice in the standard algorithms). We are now going to make some sense of (7), as follows. By the definition of the division-with-remainder of 78 by 4, we want a whole number $q$ and a whole number $r$ so that

$$78 = (q \times 4) + r \quad \text{where } 0 \le r < 4 \qquad (8)$$

We are going to estimate what $q$ must be. It cannot be a 3-digit number because the smallest 3-digit number is 100, and if $q$ has 3-digits, then the right side of (8) ≥ 400, which would contradict (8). Next, we try letting $q$ be a 2-digit number. If $q \ge 20$, then the right side of (8) would be ≥ 80, again impossible. So $q < 20$. Therefore, let $q = 10 + b$ where $b$ is a single-digit number. Then $q \times 4 = 40 + 4b$. By (8), we have $78 = 40 + 4b + r$, which gives

$$(78 - 40) = 4b + r \qquad (9)$$

This explains the appearance of 78 – 40 = 38 in (7). Next, we estimate what $b$ should be. According to (9), $r = 38 - 4b$, and since $0 \le r < 4$ by (8), we have $0 \le 38 - 4b < 4$. At this point, a knowledge of the multiplication table immediately gives $b = 9$, so that $q = 10 + b = 19$. Thus, (9) gives $38 = 36 + r$, or

$$\mathbf{38} - \mathbf{36} = r$$

On the one hand, this explains the appearance of 38 – 36 = 2 in (7). On the other hand, we get $r = 2$. Referring back to (8), we have arrived at

$$78 = (19 \times 4) + 2$$

and this is exactly (6). We have finally made some mathematical sense of the "division house" in (7) or (1) as well as its kinship to (6). In a fourth grade or fifth grade classroom, one should use the same strategy to do a few more specific examples, e.g., why the long division of 138 by 5 leads to 138 = (27 × 5) + 3, or why the long division of 781 by 4 leads to 781 = (195 × 4) + 1. The latter example will be particularly illuminating to students because they get to see that the long division of 78 by 4 in (1) is completely embedded in the long division of 781 by 4.

$$\begin{array}{r} 195 \\ 4\overline{)781} \\ \underline{4\phantom{81}} \\ 38\phantom{1} \\ \underline{36\phantom{1}} \\ 21 \\ \underline{20} \\ 1 \end{array} \qquad (10)$$

## Teaching Long Division in Grade 6

The teaching of the long division algorithm usually spans grades 4-6. We now describe what students should learn about the algorithm by the end of the sixth grade: they should know why the long division of a two-digit number by a single-digit number—such as 78 by 4—leads directly to a division-with-remainder such as 78 = (19 × 4) + 2

First of all, what is an *algorithm*? This word is used frequently in elementary school, yet it is hardly ever explained and even more rarely taken seriously in teaching. If the teaching of the standard algorithms would include an explicit description in each case of what the *algorithm* in question is (e.g., Wu, 2011, pp. 63, 74, 86-87, 108-109), then the mathematical quality of the teaching would most likely improve, as we will try to demonstrate with the long division algorithm. For school mathematics, we may define an **algorithm** to be a finite sequence of precise instructions for carrying out specific computations to result in the desired outcome at the end. To describe the *long division algorithm*, we should write down abstractly a finite sequence of steps so that, for any pair of whole numbers $m$ and $n$ ($n \neq 0$) , these steps will lead to the division-with-remainder of $m$ by $n$ in the form of (5). Something approximating this can be found in Wu (2011), Section 7.3. In a sixth-grade classroom, however, such an approach would be impractical. Instead, we will explicitly describe such a finite sequence of instructions for specific cases. For example, here is the long division algorithm of 78 by 4. An overall comment is that each step in this sequence is a division-with-remainder whose divisor is always 4 and whose dividend will involve *one digit of the dividend 78 at a time.*

**Step 1.** Perform the division-with-remainder so that its dividend is the leftmost digit 7 of 78. (Recall: its divisor is always 4.)

$$7 = (1 \times 4) + 3 \qquad (11)$$

**Step 2.** Perform the division-with-remainder so that its dividend is the sum of the next digit of 78 (which is 8) and 10 times the remainder of the preceding division-with-remainder (which is 3). (Recall: the divisor is always 4.)

$$38 = (9 \times 4) + 2 \qquad (12)$$

**Step 3.** The quotient of the division-with-remainder of 78 by 4 is obtained by "stringing together" the single-digit quotients in Steps 1 to 2, namely, 1 and 9. The remainder of the division-with-remainder of 78 by 4 is the remainder of the last step (Step 2), which is 2.

One must convince sixth graders that, strange as Steps 1-3 may seem, the long division in (1), upon closer inspection, is nothing more than a schematic representation of Steps 1 and 2. What we want to show is that *the long division algorithm is correct,* i.e., we have to prove the following theorem.

**Theorem 1.** *Steps 1 and 2 imply Step 3.*

One may think that Theorem 1 is a waste of time because to show Step 3 is correct, all we have to do is check that 78 = (19 × 4) + 2 is correct. But the theorem says much more: it says that Step 3 can be *derived strictly from Steps 1 and 2*. Thus it is more than a numerical statement that (19 × 4) + 2 is equal to 78. Rather, it asserts that we can use reasoning alone to get to the equality 78 = (19 × 4) + 2 by making use of Steps 1 and 2.

**Proof of Theorem 1.** As in the proofs of the validity of all the standard algorithms, the key ingredient is *the expanded form of a whole number* (see Wu 2011, p. 20):

$$78 = 70 + 8 \qquad (13)$$

Now, from (11), we get $7 = (1 \times 4) + 3$. Therefore,

$$70 = (10 \times 4) + 30$$

Substituting this value of 70 into (13), we get
$78 = (10 \times 4) + 30 + 8$, which is equal to

$$78 = (10 \times 4) + 38$$

(Observe that this corresponds to the subtraction $78 - 40 = 38$ in the "division house" (1).) Substituting the value of 38 in (12) into the right side of the preceding equation, we obtain

$$78 = (10 \times 4) + (9 \times 4) + 2$$

Applying the distributive law to the first two terms on the right side, we get $78 = (19 \times 4) + 2$. This shows that the division-with-remainder of 78 by 4 has quotient 19 and remainder 2, exactly as claimed by Step 3. Theorem 1 has been proved.

The first question we must ask is how this theorem and its proof are superior to the above informal argument presented in connection with equations (8) and (9). The answer is that insofar as the long division algorithm is an *algorithm*, we are duty bound to give an explicit description of every step of the algorithm, and this the earlier informal argument failed to do. Precision and clarity matter in mathematics. Moreover, mathematics is about the deduction of conclusions from assumptions, and the preceding theorem and its proof present a textbook case of this deduction process. By comparison, one is left uncertain about the precise assumptions that were made in the earlier argument. Also, see the comment following the third remark on this page.

The preceding proof should also be supplemented by three additional remarks. *First,* the long division algorithm exemplifies the recurrent theme of the standard algorithms, which is to *break up a multi-digit computation into computations involving single digits* (e.g., Wu, 2011, Chapter 3). Thus, each of Steps 1 and 2 essentially (though not literally) computes with the digits of the dividend 78 *one at a time,* and more importantly, the algorithm itself computes the quotient 19 *one digit at a time* (see Step 3). It may also be observed that although each of Steps 1 and 2 is itself a division-with-remainder, it differs from the original division-with-remainder of 78 by 4 in that the dividend in each of Step 1 and Step 2 (7 and 38, respectively) is smaller than the original dividend of 78. While this fact may not seem to be much of an advantage when the original dividend (such as 78) is relatively small, the advantage will become more pronounced as the dividend gets larger. Our next example of the long division of 781 by 4 will give a better idea in this regard.

A *second remark* is that, to the extent that there should be one general *long division algorithm* that is applicable in all cases, one may not be able to discern from the preceding Steps 1-3 what the general long division algorithm should look like. However, this lack of clarity will disappear in our next two examples with a dividend of three digits. For a more precise description of the general case, see Chapter 7 of Wu (2011).

A *third remark* is that one should take note of the fact that the algorithm, as stated in Steps 1-3, completely ignores the place value of the digits of the dividend. This fact will be more forcefully brought out after we discuss the division-with-remainder of 781 by 4. Contrary to the emphasis placed by the education literature on the concept of place value in discussing the standard algorithms, a main selling point of the standard algorithms is the *mathematical simplicity* of their execution because these algorithms intentionally ignore place value (e.g., Wu, 2011, pp. 59, 66, 120-121). Place value becomes relevant only when we try to prove that an algorithm is correct. From this perspective, the argument in connection with equations (8) and (9) is unsatisfactory because it does not draw a sharp line between the place-value independence of the algorithm itself and the key role place value plays in the justification of the algorithm.

We should also mention that there is a subtle issue involving the implicit assumption in Steps 1 and 2 that the quotient in each division-with-remainder of (11) and (12) will be a *single-digit* number. We refer the reader to Section 7.6 of Wu (2011) for the simple explanation.

As promised, we will next take up the division-with-remainder of 781 by 4. In a typical sixth grade classroom, this example would be optional, though highly desirable.

Recall first of all that no thinking is needed for getting the division-with-remainder of 781 by 4: count all the multiples of 4 up to 781. However, this is clearly a tedious process and a shortcut is called for (the tedium would be even more obvious if the dividend is not 781 but 781234). As we mentioned earlier, the long division algorithm is the sought-for shortcut. Let us first recall the long division in (10):

$$
\begin{array}{r}
195 \\
4\,\overline{)\,781} \\
\underline{4\phantom{00}} \\
38\phantom{0} \\
\underline{36\phantom{0}} \\
21 \\
\underline{20} \\
1
\end{array}
\qquad (14)
$$

It is easy to verify that this "division house" is merely a schematic representation of Steps 1-3 of the following **long division algorithm of 781 by 4:**

**Step 1.** Perform the division-with-remainder so that its dividend is the leftmost digit 7 of 781. (Recall: its divisor is always 4.)

$$7 = (1 \times 4) + 3 \qquad (15)$$

**Step 2.** Perform the division-with-remainder so that its dividend is the sum of the next digit of 781 (which is 8) and 10 times the remainder of the preceding division-with-remainder (which is 3). (Recall: the divisor is always 4.)

$$38 = (9 \times 4) + 2 \qquad (16)$$

**Step 3.** Perform the division-with-remainder so that its dividend is the sum of the next digit of 781 (which is 1) and 10 times the remainder of the preceding division-with-remainder (which is 2). (Recall: the divisor is always 4.)

$$21 = (5 \times 4) + 1 \qquad (17)$$

**Step 4.** The quotient of the division-with-remainder of 781 by 4 is obtained by "stringing together" the single-digit quotients in Steps 1-3, namely, 1, 9, and 5. The remainder of the division-with-remainder of 781 by 4 is the remainder of the last step (Step 3), which is 1.

What we want to prove is that the long division algorithm of 781 by 4 is correct, i.e., we have the following theorem.

**Theorem 2.** *The preceding Steps 1-3 imply Step 4.*

**Proof of Theorem 2.** Having gone through the proof of Theorem 1 in detail, we will be briefer this time around. As always, we begin with the expanded form of 781:

$$781 = 700 + 80 + 1 \qquad (18)$$

From (15), we get

$$700 = (100 \times 4) + 300$$

Substituting this value of 700 into (18), we have
$781 = (100 \times 4) + 300 + 80 + 1$, or

$$781 = (100 \times 4) + 380 + 1 \qquad (19)$$

Now (16) implies that $380 = (90 \times 4) + 20$. If we substitute this value of 380 into (19), we obtain

$$781 = (100 \times 4) + (90 \times 4) + 20 + 1$$

Applying the distributive law to the first two terms on the right side, we get

$$781 = (190 \times 4) + 21$$

Now substituting the value of 21 in (17) into the right side, we obtain

$$781 = (190 \times 4) + (5 \times 4) + 1$$

Using the distributive law again on the right side, we finally arrive at

$$781 = (195 \times 4) + 1 \qquad (20)$$

Since this is exactly the statement of Step 4 above, i.e., the division-with-remainder of 781 by 4 has quotient 195 and remainder 1, the proof of the theorem is complete.

### Remarks

1. Now, it should be clear from the repetitive nature of the preceding Steps 2 and 3 how the long division algorithm will proceed in the general case: Begin with the leftmost digit of the dividend as in Step 1 above and repeat the following process until you get to the rightmost digit of the dividend:

   For the dividend of the next division-with-remainder, add 10 times the remainder of the preceding division-with-remainder to the next digit to the right in the original dividend.

   Moreover, it is equally clear how to prove that the algorithm is correct: start with the expanded form of the original dividend and replace each term in the expanded form by each of the divisions-with-remainder given by the steps of the algorithm.

2. Looking back over our work so far, we can see more clearly the purpose of the long division algorithm: it is to replace the original division-with-remainder by a succession of simpler divisions-with-remainder in each of which the dividend is smaller than the original one. Thus, in the case of the division-with-remainder of 781 by 4, the dividends in (15)-(17) are 7, 38, and 21; each is far smaller than 781.

3. We are also in a better position now to understand the statement that the long division algorithm *ignores place value.* Let us compare the two long divisions: 78 by 4 and 781 by 4. The number 7 is the tens digit in 78 but is the hundreds digit in 781, yet the first steps of the algorithm in the two cases, (11) and (15), are identical. Similarly, the number 8 is the ones digit in 78 but is the tens digit in 781, and yet the second steps of the algorithm in the two cases, (12) and (16), are again identical. These confirm a key fact about the long division algorithm: it only looks at each digit of the dividend but not its place value. (Let it be said one more time that the proof of the validity of the algorithm does take into account the place value of each digit of the dividend.)

To consolidate our gains, we will take up the division-with-remainder of 242 by 16 (this is the division-with-remainder suggested in Green, 2014). Again, in a typical sixth grade classroom, this would be optional, though extremely instructive. The new feature here is that the divisor is a two-digit number. In this case, the **long division algorithm of 242 by 16** is the following:

**Step 1.** Perform the division-with-remainder so that its dividend is the leftmost digit 2 of 242 (recall: its divisor is always 16):

$$2 = (0 \times 16) + 2 \qquad (21)$$

**Step 2.** Perform the division-with-remainder so that its dividend is the sum of the next digit of 242 (which is 4) and 10 times the remainder of the preceding division-with-remainder (which is 2). (Recall: the divisor is always 16.)

$$24 = (1 \times 16) + 8 \qquad (22)$$

**Step 3.** Perform the division-with-remainder so that its dividend is the sum of the next digit of 242 (which is 2) and 10 times the remainder of the preceding division-with-remainder (which is 8). (Recall: the divisor is always 16.)

$$82 = (5 \times 16) + 2 \qquad (23)$$

**Step 4.** The quotient of the division-with-remainder of 242 by 16 is obtained by "stringing together" the single-digit quotients in Steps 1-3, namely, 0, 1, and 5. The remainder of the division-with-remainder of 242 by 16 is the remainder of the last step (Step 3), which is 2.

Here is the "division house" of the long division of 242 by 16:

```
      0 1 5
 1 6 ) 2 4 2
      0
     ------
      2 4
      1 6          (24)
     ------
        8 2
        8 0
       ----
          2
```

It is easy to see that this "division house" is nothing but a schematic presentation of the preceding Steps 1-3. Let us prove once again that the algorithm is correct.

**Theorem 3.** *In the long division algorithm of 242 by 16, Steps 1-3 imply Step 4.*

**Proof.** The expanded form of 242 reads:

$$242 = 200 + 40 + 2 = 240 + 2 \qquad (25)$$

Since the equality (21) is the trivial statement that 2 = 2, we begin with (22), which implies that $240 = (10 \times 16) + 80$. Substituting this value of 240 into (25), we get

$$242 = (10 \times 16) + 80 + 2 = (10 \times 16) + 82$$

Now substituting the value of 82 in (23) into the preceding equation, we obtain

$$242 = (10 \times 16) + (5 \times 16) + 2$$

Applying the distributive law to the first two terms on the right side, we get

$$242 = (15 \times 16) + 2$$

which is exactly the statement that the division-with-remainder of 242 by 16 has quotient 15 and remainder 2, i.e., Step 4 is correct. The theorem is proved.

Again, the repetitive nature of Steps 2 and 3 helps to give a clear conception of what the general long division algorithm is about. This *long division algorithm of 242 by 16* also serves to better highlight a special feature of the long division algorithm in general, which is to break up the original division-with-remainder of 242 by 16 into more manageable divisions-with-remainder, each with the same divisor 16, but with a far smaller dividend: the division-with-remainder of 24 by 16, and the division-with-remainder of 82 by 16. One more thing that is noteworthy is that TSM teaches the long division of 242 by 16 by saying that, since 16 does not go into 2, one should consider the first two digits 24 of 242 as the first dividend. However, the long division algorithm of 242 by 16—being an algorithm—does not depend on this contingent kind of judgment to "skip a step" in certain situations. Its instruction to perform a division-with-remainder in Step 1 is meant to be carried out literally, as it was in the equality (21).

## Curricular Implications

Because of the lack of space, we will be brief in explaining how TSM's mangling of the concept of division-with-remainder has pernicious repercussions later in the school mathematics curriculum.

The concept of the GCD [greatest common divisor; commonly referred to as greatest common factor (GCF) in school mathematics] of two nonzero whole numbers is a staple of elementary school mathematics, but TSM's failure to correctly teach division-with-remainder has forced the teaching of gcd to be confined entirely to an inspection of the factors of each number. In particular, this failure results in the Euclidean algorithm not being taught in K-12 as an effective method of getting the gcd. While we are not strongly advocating here that the Euclidean algorithm be taught in K-12, we can nevertheless amplify the fact that, by not teaching division-with-remainder properly, TSM hampers students' future mathematics learning. Consider, for example, a favorite activity in the learning of fractions: how to simplify the following fraction to lowest terms:

$$\frac{551}{247}$$

It is not so easy to factor either 551 or 247, but if we do the long division of 551 by 247, we get

$$551 = (2 \times 247) + 57$$

It is a fairly straightforward consequence of this equality that the two pairs {551,247} and {247,57} have exactly the same collection of common divisors (e.g., Wu, 2011, p. 465; Wu, 2016a, p. 210). But the second pair has the advantage that 57 is considerably smaller than 247 or 551, and therefore, the search for the gcd of {247,57} promises to be potentially easier than that of {551,247}. As a matter of fact, it is so easy to get all the factors of 57 (they are 1, 3, 19, and 57 because $57 = 3 \times 19$) that we know the common divisors of {247,57} are among 1, 3, 19, and 57. It is then painless to conclude that the gcd of {247,57} is in fact 19, so that 19 is also the gcd of 551 and 247. Hence,

$$\frac{551}{247} = \frac{19 \times 29}{19 \times 13} = \frac{29}{13}$$

The idea that the task of finding the gcd of two whole numbers can be simplified by just one application of long division—without a doubt—deserves to be taught, but it *cannot be taught* if all that TSM has to offer about long division is of the "551 ÷ 247 = 2 R57" variety.

Needless to say, the Euclidean algorithm is just an iteration of the preceding process (e.g., Wu, 2011, pp. 464ff.; Wu, 2016a, pp. 203ff).

Of course, the numbers in the preceding problem have been rigged to heighten the drama (!), but the message should not be lost, to the effect that the long division algorithm is not just a boring arithmetic skill but is a very versatile mathematical tool (e.g., Wu, 2011, Part 4). More importantly, such an application of the long division algorithm is a powerful illustration of the coherence of mathematics: the fact that there are hidden connections between seemingly disparate topics, e.g., long division and simplifying fractions. TSM routinely makes it impossible for students to see this and other hidden connections.

The issue of the coherence of mathematics naturally brings us to two other topics in the school curriculum related to long division. First, the division algorithm for polynomials of one variable states that given two such polynomials $F(x)$ and $G(x)$ (with $G \neq 0$), $F$ can be expressed in terms of $\boldsymbol{G}$ and two polynomials $Q(x)$ and $R(x)$ as follows:

$$F(x) = Q(x)G(x) + R(x) \quad \text{where } \deg R(x) < \deg G(x)$$

(see, e.g., Section 5.1 of Wu, 2020b). This is clearly an analogy of division-with-remainder, with $Q$ and $R$ playing the roles of **quotient** and **remainder**, respectively, and $F$ and $G$ playing the roles of **dividend** and divisor, respectively. This is not quite a direct generalization from whole numbers to polynomials. The difference is that, for the division-with-remainder, the comparison of the remainder to the divisor is by using the magnitudes of the whole numbers, but the comparison in the case of the division algorithm for polynomials is by using the *degrees* of the polynomials. Thus, polynomial division is reminiscent of long division but not a generalization of it. In advanced mathematics, both will become special cases of *Euclidean domains*—again, a reminder of the coherence of mathematics. But since TSM does not teach division-with-remainder, there can be no such intellectual resonance when students come to the division algorithm for polynomials in Algebra II. This is an opportunity wasted.

Our next topic is about the conversion of a fraction to a decimal "by the long division of the numerator by the denominator." This is such a well-known topic in middle school mathematics that it suffices to use a simple example to illustrate the fundamental ideas involved. We claim:

$$\frac{7}{16} = 0.4375 \qquad (26)$$

where 4375 is the quotient of the *long division* of $7 \times 10^4$ by 16. Here, the validity of the equality of the two numbers in (26) is not in question because, by the definition of a decimal (e.g., Wu, 2011, p. 187),

$$0.4375 = \frac{4375}{10{,}000}$$

and the fraction on the right easily simplifies to $\frac{7}{16}$. What is at issue, rather, is why the fraction $\frac{7}{16}$ is equal to the quotient resulting from the long division of $7 \times 10^4$ by 16, but with the decimal point inserted in front. (Just as in Theorem 1, we have again an example of the emphasis placed on the method of arriving at a conclusion rather than on the validity of the conclusion itself.) This issue—that the fraction is equal to the decimal produced by the long division—is precisely what is not addressed in the usual educational discussions, which are usually preoccupied with the "repeating" property of the resulting decimal. Therefore, what we seek is not just a proof of (26) per se, but an explanation for the intrusion of long division into this conversion procedure and how the decimal point materializes. To this end, we will prove something more general: let the long division of $7 \times 10^k$ ($k$ being any nonzero whole number) by 16 have quotient $N$ and remainder $r$, then

$$\frac{7}{16} = \frac{N}{10^k} + \frac{r}{16 \times 10^k} \qquad (27)$$

First, we show that (27) implies (26). Let $k = 4$ in (27). Since $7 \times 104 = 4375 \times 16$, we see that $N = 4375$ and $r = 0$ in this case. Therefore (26) immediately follows. Next, to prove (27), we have by the definition of $N$ and $r$ that

$$7 \times 10^k = (N \times 16) + r$$

Therefore,

$$\frac{7}{16} = \frac{7 \times 10^k}{16 \times 10^k} = \frac{(N \times 16) + r}{16} \times \frac{1}{10^k} = \frac{N}{10^k} + \frac{r}{16 \times 10^k}$$

The proof of (27) is complete. We also pause to note that included in (27) is the assertion that if we let k = 5 or any whole number > 4, (27) will still imply (26).

It is not out of place to remark that the simplicity of the preceding reasoning comes from the clear understanding of how long division leads to division-with-remainder as in Theorems 1-3. We should also point out that (27)—together with its proof above—remains valid, verbatim, when the fraction $\frac{7}{16}$ is replaced by any fraction. This is the key step that leads to the general proof of the fact that the conversion of a fraction to a decimal can be achieved "by the long division of the numerator by the denominator." For the details, see Section 3.4 of Wu (2020c). (A simplified version without the use of limits has been given in Chapter 42 of Wu, 2011.)

The conversion of a fraction to a decimal by long division is a staple of school mathematics, and we take for granted that someone must have written down a proof of the correctness of this procedure. Yet, it seems almost impossible to locate a correct proof in the recent education literature. So long as TSM rules over school mathematics and so long as it preaches that $16 \div 7 = 2\ R2$, there can be no hope for such a proof. Once again, the surprising connection between long division and the conversion of a fraction to a decimal is lost because TSM has rendered mathematical reasoning impossible.

## Must We Eradicate TSM?

In her popular article, Elizabeth Green (2014) makes a passing comment on the traditional way of teaching the "division house" as nothing more than a ritualistic inculcation of mind-numbing procedures that turns "school math into a sort of arbitrary process wholly divorced from the real world of numbers." Green's article suggests that the road to improvement is to change the *teaching* of school mathematics by tapping "into what students already understood and then [building] on it.

...By pushing students to talk about math," they will uncover their own misunderstandings about division and make sense of the "division house."

We prefer to take a simpler, more grounded view on the rampant non-learning of school mathematics in the past decades. When teachers are taught only TSM, they will naturally teach only TSM to their students. The ritualistic inculcation of procedures in school mathematics classrooms is mostly—though not totally—a reflection of teachers' content-knowledge deficit. But teachers are not to blame for the sorry spectacle—the education establishment is. Why should we expect teachers to help students understand school mathematics when we have made no effort to help teachers understand it? It is altogether unrealistic to expect teachers to uncover by themselves the substantial mathematical content that undergirds the "division house" (see Theorem 1-3 above). All of us in the education community share the responsibility for helping teachers shed their baggage of TSM and acquire a new and correct knowledge base for teaching, but on this job, we have thus far fallen flat on our faces. Once we succeed in providing teachers with the mathematical knowledge they need for teaching, then it would be realistic to consider how teachers could better teach school mathematics.

The main contention of this article is that TSM is destroying school mathematics education. Now, one does not make such a sweeping statement unless one has incontrovertible proof. In this case, the proof of how destructive TSM really is can be easily accessed everywhere: pick up any of the standard school mathematics textbooks published between 1970 and 2015 or, in all likelihood, if you pick up any of the current textbooks and open it up to a random page, your chance of witnessing TSM at work is likely to exceed 80%. It is not possible in the year 2020 to have a meaningful discourse on improving school mathematics education without first initiating some serious attempts to get rid of TSM.

This article has chosen to focus on one topic, long division, to reveal the truly anti-mathematical nature of TSM. In our extended discussion, we have shown how TSM has perverted the mathematics of this single topic and made it unlearnable. But of course, we can say the same about almost every topic in the mathematics of K-12. If we want to change course and make school mathematics learnable, we must have the necessary political will to make the change and the political muscle to implement this change. But what has been long overlooked is that, besides the will and the muscle, there must also be a detailed version of correct and learnable school mathematics from K to 12—all of it —that is universally available to provide guidance. It is the absence of such an alternate version that has shipwrecked every reform since the 1950's (e.g., Wu, 2020d, pp. 7-10 ). The author has just completed a first attempt at presenting such an alternate version in the form of six volumes: Wu (2011) for elementary school, Wu (2016a) and Wu (2016b) for middle school, and Wu (2020a), Wu (2020b), and Wu (2020c) for high school. In such a large undertaking (the six volumes comprise about 2,500 pages), some topics, no matter how significant, will inevitably get short shrift, and the long division algorithm is one such example. This article may therefore be regarded as one of many needed supplementary commentaries on these six volumes. We hope that it will also help raise awareness of how pervasive the damage done by TSM has been.

It is quite common to hear people say, "I am not good at math," without a trace of self-consciousness or embarrassment. What they actually mean is something like "I am no good at memorizing an unending collection of meaningless factoids and bags of tricks for getting answers (i.e., TSM)." Indeed, no one should feel self-conscious or embarrassed about not being able to learn TSM. The urgent question is: can we spare the next generation the trauma—and the inevitable punishment for failure—of a thirteen-year immersion in unlearnable math?

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# Conceptualizing Student Responsibilities in Discourse-Rich Classrooms

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**ABSTRACT** While a growing body of research examines teachers' facilitation of discourse-rich classrooms, surprisingly little research is devoted to learners' responsibilities in such classrooms. In this paper, we share a theoretical model for explaining students' responsibilities in yielding mathematical learning in discourse-rich classrooms. These responsibilities consist of the following: (1) determined listening and striving to understand others' contributions; (2) proactive contribution; (3) maintaining equal positioning; (4) willingness to resolve incommensurability; and (5) on-task talk. Each of the responsibilities is interdependent, suggesting that failure to meet one responsibility decreases the likelihood that another responsibility will be met. The model suggests important implications for supporting learners in discourse-rich classrooms.

**KEYWORDS** *discourse, mathematics, collaboration, student responsibilities, collaborative learning*

Discourse-rich, student-centered classrooms dominate current trends in mathematics education policy, research, and practice. Advocates of discourse-rich classrooms suggest that students work with classmates to solve cognitively demanding tasks, while teachers act as facilitators, guiding students to co-construct knowledge with their peers (e.g., Jackson et al., 2012; Stein et al., 2008; Van de Walle et al., 2016). Curriculum and policy documents in the U.S. (e.g., Common Core State Standards Initiative [CCSSI], 2010; National Council of Teachers of Mathematics [NCTM], 2000, 2014); Department of Education, 2014) consistently recognize the influence of discourse-rich classrooms and suggest that "mathematical discourse among students is central to meaningful learning of mathematics" (NCTM, 2014, p. 35). Further, empirical research suggests that discourse-rich classrooms increase opportunities for authentic engagement and equitable mathematical participation (Brown, 2007; Esmonde & Langer-Osuna, 2013; Jarosz et al., 2017; Summers, 2006). Still, there is much to learn regarding how discourse-rich classrooms should operate.

A growing body of research has examined classroom norms (Partanen & Kaasila, 2015; Yackel & Cobb, 1996), teaching practices (Herbel-Eisenmann et al., 2013; O'Connor & Michaels, 2019), and the role of tasks (Henningsen & Stein, 1997; Jackson et al., 2012) in discourse-rich classrooms. However, there is surprisingly little research devoted to learners' responsibilities in such classrooms. Several questions persist regarding student accountability, such as the following: How should students communicate with one another to ensure an optimal environment for learning? What communicative behaviors enhance equal participation structures in collaborative environments? How should students resolve conflict when talking about mathematics? Answering these and other related questions requires a student-level examination of discourse-rich classrooms. To provide theoretical insight into student-level factors of collaborative classrooms, it is constructive to conceptualize first what students should be doing.

In this paper, we aim to share a theoretical model explaining students' responsibilities for yielding mathematical learning in discourse-rich classrooms. The model was refined through an iterative process of generating cross-cutting themes from literature and our own experiences researching and facilitating discourse-rich classrooms in middle-grade and university settings. Taken together, we report a model that is informed by research and corroborated by experience. We make no claims regarding the exhaustive nature of our model. Rather, we identify several student-level factors that, based on our research and experience, explain how students learn in collaborative environments. In the next section, we briefly describe our conceptualization of discourse-rich classrooms through a discursive perspective on learning.

## Discourse-Rich Classrooms

While discourse-rich classrooms vary in structure, they exhibit two defining characteristics: (1) students actively participate and communicate with others, and (2) learning is presumed to occur through mutual communication. First, students in discourse-rich classrooms actively participate by communicating with their peers and teachers. Depending on the classroom structure, this may occur in a variety of ways. Stein and colleagues (2008) suggested that discourse-rich classrooms proceed in three phases: launching a mathematical task, exploring a problem in small groups, and discussing and summarizing the problem through whole-class dialogue. They summarized a typical discourse-rich classroom as follows:

> During this 'launch phase,' the teacher introduces the students to the problem, the tools that are available for working on it, and the nature of the products they will be expected to produce. This is followed by the 'explore phase' in which students work on the problem, often discussing it in pairs or small groups. As students work on the problem, they are encouraged to solve the problem in whatever way makes sense to them and be prepared to explain their approach to others in the class. The lesson then concludes with a whole-class discussion and summary of various student-generated approaches to solving the problem. (Stein et al., 2008, p. 316)

While not all discourse-rich classrooms proceed in a similar manner as envisioned by Stein and colleagues, students in discourse-rich classrooms actively communicate about mathematics rather than passively listening to the teacher.

Another defining characteristic and by-product of the first characteristic is that learning is presumed to occur through mutual communication. Proponents of discourse-rich classrooms assume that students learn collaboratively by contributing mathematical ideas and listening to others' ideas (Scardamalia & Bereiter, 2006). This is in stark contrast to teacher-centered classrooms, wherein learning is presumed to occur through passive participation.

## Mathematical Learning as Changing Discourse

In this paper, we assume that the goal of discourse-rich classrooms is to produce mathematical learning. Aligning with Sfard (2008), we define learning as a lasting change in discourse. Accordingly, communication in discourse-rich classrooms may be considered *productive* if it leads to changes in students' discourse that are *durable* and *desirable* (Sfard & Kieran, 2001). Changes in student discourse are durable if they are likely to continue in future communication, while changes in student discourse are *desirable* if they align with accepted discourse practices of the broader discourse community. Mathematical discourse is distinguishable according to four features: word use (e.g., keywords related to numbers and shapes), visual mediators (e.g., operators, coordinate plane), narratives (e.g., theorems and definitions), and routines (e.g., repetitive patterns) (Sfard, 2008). Therefore, we will consider discourse-rich mathematics classrooms to be *productive* if they often lead students to exhibit a lasting change in the way they communicate about keywords in mathematics, visual mediators, narratives, and routines.

To illustrate *productive* communication, consider a scenario wherein two learners, Aaron and James, discuss the area of a square. Aaron conjectures that the area of a square is always larger than the side length. James contradicts this assertion and shares a counterexample (e.g., side length = 0.4). After listening to James' counterexample, Aaron agrees and suggests that if the side length is between zero and one, the area is smaller than the side length. In this exchange, Aaron changed his discourse in a *desirable* way. If this change in discourse persists in future communication, we may deem Aaron and James' communication as *productive*.

For our model, we perceive Sfard and colleagues' (Sfard, 2008; Sfard & Kieran, 2001) conceptualization of mathematical learning and *productive* communication as a primary goal for discourse-rich classrooms. Therefore, our model is designed to describe student responsibilities for engaging in such communication. In the following section, we describe five student responsibilities that we find integral to promote learning in discourse-rich classrooms. Then, we explain how these five responsibilities interact to form a model of student responsibilities.

## Student Responsibilities for Discourse-Rich Classrooms

Based on prior research and our experiences as educators, we suggest five responsibilities for which students are accountable in discourse-rich classrooms to promote mathematical learning: (1) determined listening and striving to understand others' contributions, (2) proactive contribution, (3) maintaining equal positioning, (4) willingness to resolve incommensurability, and (5) on-task talk. We discuss each of these responsibilities by reviewing relevant literature and examining episodes from our research (Campbell & Hodges, 2020; Campbell & King, 2020; Campbell et al., 2020) and teaching experiences.

### Determined listening and striving to understand others' contributions

To communicate in ways that lead to learning, students must be determined to listen to peers and actively strive to understand their peers' contributions. Scholars refer to this type of engagement as aligning frames (van de Sande & Greeno, 2012), discussing proposals (Barron, 2003), and communicating effectively (Ryve et al., 2013; Sfard & Kieran, 2001). van de Sande and Greeno (2012) suggested that students working in groups align their frames by either mutually drawing on common knowledge or actively listening to other participants with relevant knowledge to complete the task. Complementing van de Sande and Greeno's (2012) study, Barron (2003) and Sfard and Kieran (2001) found students must become active rather than passive while other group members are talking to generate learning opportunities. In short, *determined listening and striving to understand others' contributions* refers to listening actively and seeking to understand others' mathematical strategies by asking clarifying questions, building off others' contributions, and using other communicative behaviors that reveal a motivation to understand.

Though active listening may seem straightforward, it requires intense determination and often does not come naturally to learners. In our research with middle school students (Campbell & King, 2020; Campbell et al., 2020), we noticed the rarity of active listening amongst students working in groups. For instance, consider the following transcript between Josh and Amber as they attempted to create an argument for the claim "the sum of two odd numbers is even." (utterances 1 – 3).

1. *Amber:* In every single-digit number, that is odd if you know they will be even added together, then adding an odd to a two-digit number that is odd, then the answer will be even like the single-digit number was.

2. *Josh:* Alright, Amber. So... Alright, so. Um, if you go back down to the basics, seven plus five um, is twelve. Yeah, it is. OK. And seven plus three is ten. And all the basic, tiny numbers—the one-digit numbers. They all equal evens, so that means, uh, because it just depends on the last number in the number, uh, to make it an even. So, since all of the one-digit numbers are even, it just comes down to the one-digit numbers in the big number. You guys get what I'm saying? Do you want to write something down?

3 *Amber:* No, you can write something down, but I don't get what you're saying. Write in good handwriting, please.

Interestingly, in this exchange, Amber and Josh shared nearly identical arguments (utterances 1 and 2). Josh did not try to connect any of Amber's ideas and, instead, presented his argument as a new strategy. Neither student asked clarifying questions nor attempted to engage with the other's idea, resulting in ineffective communication. As a result, Amber implored Josh to simply write his answer on the task sheet ("I don't get what you're saying. Write in good handwriting, please"). While not all instances of passive listening are so obvious, it seems that learners often do not actively listen to understand one another's contributions. Determined listening requires listeners to ask clarifying questions or build upon others' responses. For instance, consider the transcript below extracted from a previous research project (Campbell & Hodges, 2020) between three college-aged learners discussing the meaning of the index of terms in a set (utterances 4 – 10).

4. *Katrina:* Yeah. OK, $j$ is the set of—and $m$, or $j$—and $m$ is—would be like the index of that set.

5. *Danielle:* Is the index of $j$.

6. *Katrina:* Index of $j$, OK.

7. *Danielle:* Now, let's make sure we can all understand this before we write it down.

8. *Hayden:* I don't understand index.

9. *Danielle:* It's the little numbers.

10. *Katrina:* It like defines, yeah. So, it defines the location in the set. So, like if you said $j$ sub $m$ and you wanted to find $j$ sub 4, it would be 7 [showing example on paper].

Danielle displayed active listening by clarifying Katrina's initial proposal ("Is the index of j."). Hayden suggested that she did not understand what Katrina meant by the word index ("I don't understand index."), resulting in the group further explaining their use of the term (utterances 9-10). Danielle's suggestion, "Now, let's make sure we can all understand this before we write it down," clearly portrays a propensity for engaging with others' thoughts. While Katrina's final explanation was not mathematically complete (utterance 10), the group came to a collective understanding of how they would use indexes to define a number in a set. In this exchange, all group members were determined to listen and respond to one another. As a result, they generated opportunities to change their working mathematical definition of and discourse related to the term *index*. Productive communication is at least partially dependent on learners' abilities to actively listen for understanding while others are talking.

### Proactive contribution

The second student responsibility for discourse-rich classrooms is *proactive contribution*, which refers to learners' willingness to offer their mathematical insight while collaborating with others. Authentic participation is central to learning mathematics (Cuoco et al., 1996; Lave & Wenger, 1991), so students must actively participate by writing and sharing their problem-solving strategies. Additionally, when students proactively contribute, they allow for a diverse range of ideas to be heard, increasing learning opportunities. Barron (2000) suggested mutuality, or the potential for all group members to contribute, is essential for effective group problem-solving. In our research with college-aged learners, we found a positive relationship between mutuality and *productive* group engagement (Campbell & Hodges, 2020).

In comparison with teacher-centered classrooms, it is simple to recognize why *proactive contribution* is necessary for productive discourse-rich classrooms. In teacher-centered classrooms, a minority expert (i.e., teacher) offers most of the mathematical discourse. Similarly, in discourse-rich classrooms without *proactive contribution*, a minority of students offer most of the mathematical insight. In both situations, the majority of students are passive observers. If one is forced to participate in such a way, it seems advantageous to observe the most experienced and knowledgeable contributor. Since the teacher is the most experienced and knowledgeable contributor in most classroom settings, teacher-centered classrooms seem a preferred environment for learning compared with discourse-rich classrooms with low student participation.

### Maintaining equal positioning

*Maintaining equal positioning* is another student responsibility that is important for promoting equitable mathematical participation structures. The literature suggests that students position or label (van Langenhove & Harré, 1999) themselves and others while working collaboratively in ways that increase or inhibit opportunities for authentic mathematical participation (Barnes, 2004; Bishop, 2012; Campbell & Hodges, 2020; Wood, 2013; Wood & Kalinec, 2012). Often, this act of positioning is tacit, while other times, learners purposely create hierarchies while working with others (van Langenhove & Harré, 1999). Barnes (2004) found evidence for fourteen different positions assumed by eleventh-grade learners in Introductory Calculus classrooms. Learners assumed positions of expert, audience, manager, helper, and so on. Some of the positions, such as the position of expert, allowed students to participate in mathematics authentically. In contrast, other positions, such as helper, relegated students to perform menial tasks without engaging in meaningful mathematics. Similarly, Bishop's (2012) analysis revealed the influence of positioning on mathematical participation. In their analysis, two seventh-grade girls constructed a hierarchy amongst themselves, resulting in one girl being labeled as smart and the other girl labeling herself as a less competent doer of mathematics. These positions influenced the roles each girl assumed while problem-solving. For instance, the girl positioned as a competent doer of mathematics often controlled problem-solving activities. Positions, whether tacitly or purposely assigned, often result in hierarchical classroom structures, privileging meaningful mathematical access for some and denying access for others.

In our work with college-aged learners, we found that students working in groups tend to assume positions that fall on a spectrum, with passive observers on one end of the spectrum, dominant controller on the other end, and balanced negotiator in the middle (Campbell & Hodges, 2020). Passive observers mostly listen to group communication, while dominant controllers dominate group discussion. Balanced negotiators both offer their contributions and actively seek to negotiate with others. Students are often forced into passive observer positions by other group members who dominate group discussions. Conversely, students are sometimes forced into dominant controller positions if other group members refrain from contributing to group conversations. In short, each group member's positioning influences mathematical participation. For collaborative engagement to be *productive*, learners should seek to position one another as equals. For instance, students should take turns assuming positions of *expert* (Barnes, 2004) or other positions that provide access to participation. Additionally, learners should actively seek to monitor their positions and realize when they become dominant or passive. By maintaining equal positioning and reflecting on their positions, students create equitable participation structures in discourse-rich classrooms

### Willingness to resolve incommensurability

Incommensurability, or conflicting discursive rules related to a similar topic, often occurs between learners working in a collaborative environment (Sfard, 2019). For instance, in the hypothetical scenario presented in a previous section ("Mathematical Learning as Changing Discourse"), Aaron and James suggested two mathematical strategies that contradict one another. When two or more participants exhibit incommensurable discourses related to a similar topic, they may resolve the conflict through discussion, argue without a resolution, or avoid conflict altogether. Avoiding conflict altogether reduces opportunities for learning since students do not experience opportunities to change their discourse if their ideas are unchallenged. Instead, to engage in *productive* communication, students must be determined to resolve incommensurability by discussing opposing strategies (Chiu, 2000, 2008a, 2008b; Jarosz et al., 2017; Orme & Monroe, 2005; Sfard, 2007, 2019) studies with ninth-grade Algebra students revealed that polite disagreements, or respectful arguments about mathematical strategies, were significantly positively correlated with success and creativity in group problem-solving. However, rude disagreements were negatively correlated with group success, indicating that the nature of conflict, whether polite or rude, influences the likelihood of resolving incommensurability. Other studies similarly corroborate the influence of argumentation on learning (e.g., Jarosz et al., 2017).

In our work with middle-school and college-aged learners, we have come to learn that students are often reluctant to resolve incommensurability. Instead of arguing about the viability of their approaches, they often 'agree to disagree' or refrain from engaging in conflict at all. For instance, consider the following exchange between two middle-grade learners who were working in a group to construct an argument for the claim 'the sum of two odd number equals an even number' (utterances 11 – 12; data obtained in a prior research project [Campbell & King, 2020]).

11. *Brittany:* I just like added all the odd numbers 1 – 9, and they all became even because they're all divisible by 2. And all numbers end with a 1 – 9, so if it's odd, then it will, then you just add it with another odd number, and it's divisible by 2.
12. *Felicia:* Yeah, that's...OK. So, I'll write out what I put out, you write out what you put, and then you write out whatever you put.

During this exchange, instead of discussing their arguments to decide which was viable or most efficient, Felicia suggested the group simply compile all their strategies on the task sheet (utterance 12). From our experience, this avoidance of critique and argumentation is evident across the grade levels.

While students often avoid conflict, there are times that they willingly seek to resolve incommensurability. Danielle, Katrina, and Hayden's (three college-aged students) interaction on a mathematical proving task portrays the benefits of deliberating about the viability of differing mathematical strategies (utterances 13-18; data obtained in a prior research project [Campbell & Hodges, 2020]).

13 *Katrina:* We multiply 3, 5, and 7 just for kicks. It creates 105, and when you make—which it sounds dumb. Would we be able to talk about 105 in terms of what it means to be prime? Because then the only factors of 105 will then be 3, 5, and 7. I don't know if it's helpful.

14 *Hayden:* That's not the only factors. Those are the prime factors.

15 *Katrina:* Those are the only ones.

16 *Danielle:* 21 is also a factor of 105.

17 *Katrina:* Well, yeah, but then that factors out to be 3 and 7.

18 *Danielle:* Right, which would make them prime factors.

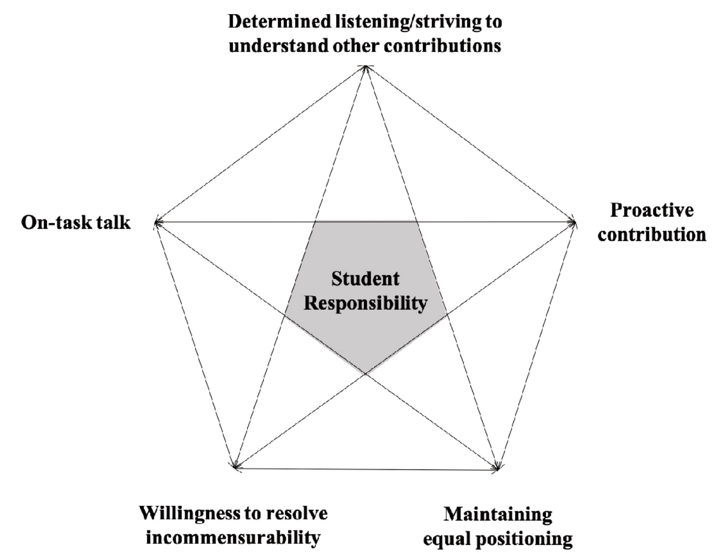
The group deliberated about Katrina's claim that 3, 5, and 7 were the only factors of 105. By deliberating with one another, the group came to the understanding that 3, 5, and 7 were the only *prime* factors of 105—not the only factors, as Katrina originally suggested. As evidenced by this interaction, willingly engaging in conflict to resolve incommensurability provides students with opportunities to change their discourse in desirable ways.

### On-task talk

The final student-responsibility is *on-task talk*. For discourse-rich classrooms to be *productive*, students are responsible for ensuring the majority of their talk is related to mathematics. Some researchers found occasional off-task communication can aid non-dominant students in gaining power and agency in mathematics classrooms (e.g., Esmonde & Langer-Osuna, 2013). Without disregarding such findings, empirical research also suggests that *on-task talk* is highly predictive of successful collaborative problem-solving (Chiu, 2008; Jarosz et al., 2017). For instance, Jarosz et al. (2017) investigated predictors of successful group problem-solving for college-aged learners in an introductory statistics course. They found that successful groups utilized a lower proportion of off-task talk than less successful groups. Research does not suggest that groups should never engage in off-task talk. Indeed, such environments may become unauthentic or unenjoyable for students. Rather, learners should limit distractions and ensure the *majority* of their talk is related to mathematics. In the next section, we share our theoretical model of student responsibilities in discourse-rich classrooms and explain how the five responsibilities interact and influence one another.

**Figure 1**

*Theoretical Model of Student Responsibilities in Discourse-Rich Classrooms*




**Note:** This model shows the connection amongst student responsibilities in a discourse-rich classroom.

## Model of Student Responsibilities

Thus far, we have described five student responsibilities that promote learning in discourse-rich classrooms. We claim that the five responsibilities do not operate in isolation. Instead, each student's responsibility is intertwined with the others, influencing the likelihood that another responsibility may be upheld. Figure 1 shows the connection amongst the responsibilities.

The double-sided arrow signifies their interdependence. To illustrate the interdependency, consider *proactive contribution* and its relationship to the other four responsibilities. *Proactive contribution* and *determined listening and striving to understand others' contributions* are interdependent because, when few students proactively contribute, there are few opportunities to listen actively with the purpose of understanding. Likewise, if learners do not believe others are actively listening to their contributions, they will be unlikely to contribute proactively. *Proactive contribution* and *maintaining equal positioning* are interdependent since students' willingness to communicate in groups influences how they position themselves and others. Likewise, students' positions influence how compelled they feel to communicate in groups. *Proactive contribution* and *on-task talk* reveal a trivial interdependency. Finally, *proactive contribution* and *willingness to resolve incommensurability* are interdependent since students can only resolve conflict if multiple group members provide mathematical contributions. Likewise, students will only be willing to resolve conflict if there is a group norm of proactive communication amongst all group members. Indeed, each student's responsibility reveals an interdependency on other responsibilities. Such an interconnected model suggests learning in discourse-rich classrooms is an intricate process. Failure to maintain one responsibility could inhibit the potential to maintain other responsibilities, which is detrimental to creating learning opportunities in a collaborative environment. The care required for the successful implementation of discourse-rich classrooms is well-documented in the literature (e.g., Sfard & Kieran, 2001; Webel, 2013). Students must be supported in meeting their responsibilities to create learning opportunities.

## Discussion

In this paper, we offered a theoretical model revealing students' responsibilities for *productive*, discourse-rich classrooms. The model consists of five components: (1) determined listening and striving to understand others' contributions, (2) proactive contribution, (3) maintaining equal positioning, (4) willingness to resolve incommensurability, and (5) on-task talk. Each of the components are interrelated and influence one another, suggesting that neglect of one component of the model decreases the likelihood that another responsibility will be maintained.

The five student responsibilities help learners engage in *productive* communication in discourse-rich classrooms to experience lasting mathematical discourse changes (i.e., mathematical learning; Sfard, 2008). From a theoretical standpoint, the responsibilities provide learners with opportunities for their current discursive rules to be challenged, which can result in *durable* and *desirable* mathematical discourse changes (Sfard, 2008). For instance, active, determined listening and willingness to resolve commensurability promote opportunities for learners to confront others' mathematical ideas. Similarly, *on-task talk* ensures that learners' mathematical contributions remain the focal point of deliberation, which is necessary for learners to experience opportunities to change their mathematical discourse. Each responsibility creates opportunities for conflict resolution, which can lead students to change their mathematical discourse in ways that are *durable* and *desirable*.

Our model suggests several implications for future research. The model leaves room for theoretical and empirical refinement. Future research might uncover other student responsibilities that are integral for *productive* discourse-rich classrooms or might determine more precise linkages between responsibilities. For instance, while we suggest that all responsibilities are interdependent, it is possible that some linkages are stronger than others. Therefore, some responsibilities may carry more weight in determining the productivity of collaborative engagement than others. Scholars might also seek to design a pedagogical model for aiding learners in meeting their responsibilities. Current literature on teacher facilitation of discourse-rich classrooms mostly focuses on teacher moves (e.g., revoicing) that promote a positive classroom culture (O'Connor & Michaels, 2019). Scholars might extend research on teacher moves by empirically investigating strategies to aid students in meeting their responsibilities in discourse-rich classrooms.

Another extension of our work is the consideration of student responsibilities for meeting different goals in discourse-rich classrooms. Based on Sfard and Kieran's (2001) conceptualization of productivity, we defined discourse-rich classrooms as *productive* if they often lead students to change their discourse in *durable* and *desirable* ways. This is an important outcome of education, but it does not capture all the potential goals of social perspectives on learning. For instance, some scholars suggest that discourse-rich classrooms help learners engage in important social skills such as argumentation and explanation (Hmelo-Silver et al., 2007). The proposed model does not take into consideration other potentially important outcomes of discourse-rich classrooms. Future research might expand, combine, or create new models of student responsibilities for meeting various goals.

In relation to practice, teachers might explicitly teach learners their responsibilities for maintaining a *productive* learning environment. However, unlike other strategies for improving teaching and learning, our model should be considered a whole unit. That is, it may be unproductive for learners to practice responsibilities one after another until mastery is reached. The five responsibilities work in tandem, and increased maturity in one responsibility is likely to enhance other responsibilities. Therefore, we suggest that practitioners introduce students to their responsibilities and work as a community towards maturation. Strategies for teaching the responsibilities and making them normative in a classroom community are beyond the scope of this paper. Still, reflection seems a promising tool for increasing student awareness of their actions (Wagner, 2007). By continually reflecting on their progress, students might become more aware of their abilities to meet their responsibilities for discourse-rich classrooms.

In closing, the field still has much to learn regarding how discourse-rich classrooms *should* operate. Current research is unbalanced, with most studies examining teacher facilitation while placing little emphasis on student-level factors. To understand supportive actions in discourse-rich classrooms, the field might further examine how students communicate with their peers and teachers. Analyzing such communication from the student-level can reveal desirable or undesirable communicative behaviors, suggesting further implications for pedagogical design. This paper might act as a starting point for future empirical analyses of student-level research in discourse-rich classrooms.

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# The Mathematical Mindsets and Mathematical Identities Revealed in Social Media Discourse

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**ABSTRACT** Mathematics problems are shared rapidly across all social media platforms, and the relative anonymity granted to users can lead to unfiltered discourse. This study examined 1,046 comments from a mathematics problem posted twice to YouTube in February 2016 to determine the underlying narratives that indicate the commenters' mathematical mindsets and their mathematical identities. These two factors contribute to mathematics success in general. Qualitative themes emerged regarding attributions, motivational goals, response to failure, defensive processing, normative comparisons, and positional acts. A fixed mathematical mindset was the dominant mindset and corresponded to positional acts of superiority, inferiority, or authority. This finding suggests that intellectual capacity or ranking was a core component of the mathematical identities for these users. The growth mathematical mindset was linked to spectator and instructor or solidarity positions, suggesting that these users had more robust mathematical identities that were unthreatened by performance indicators. Further examination of social media discourse and its relation to mathematical mindsets and mathematical identities can lead to a better understanding of the interactions outside the classroom that either encourage or inhibit mathematics success.

**KEYWORDS** *implicit theories of intelligence, mindsets, mathematical mindsets, mathematical identity, discourse, positional acts, social media*

## Overview

### Social Media

In January 2020, 4.5 billion people used the internet; of those users, 3.8 billion engaged in social media (Nazir & Dubras, 2020). The ubiquity of social media to daily life has resulted in digital footprints that are increasingly intertwined with social interactions that can render both beneficial and harmful changes to mental well-being. For instance, social media can beget positive health effects when it is used to facilitate actions that increase our social capital:

> Individuals who are members of a social network, as opposed to those who are not, have access to information, social support, and other resources such as other network members' skills and knowledge due to their network membership or social connections. (Bekalu et al., 2019, p. 69S – 70S)

As a result, some social media users have improved their mental health. For example, social media users have reported feeling a stronger sense of community and being more emotionally supported (Royal Society for Public Health, 2017). However, social media usage can also lead to harmful consequences by increasing adverse health effects, such as anxiety, depression, and poor sleeping patterns. In May 2017, the Royal Society for Public Health and the Young Health Movement surveyed 1,500 people aged 14-24 in the UK and found that four of the five most used social media platforms for their age demographic (Facebook, Instagram, Twitter, Snapchat, and YouTube) increased their feelings of anxiety and depression (Royal Society for Public Health, 2017). Of note, YouTube was the only social media platform to have shown a positive effect in this respect.

## Mathematical Discourse and Implicit Theories of Intelligence

The relative anonymity afforded to social media users contributes to discourse that is often unfiltered (i.e., audacious and communicated without consideration to the audience). So, what happens when you introduce an innocuous mathematics problem to this nearly unbridled comment culture? On its own, mathematics discourse can be enlightening. Our response to a mathematics problem can manifest our implicit theory of intelligence, which Hong et al. (1995) describes as:

> Beliefs about the fundamental nature of intelligence, specifically whether intelligence is a fixed entity that cannot be changed (an entity theory) or a malleable quantity that can be increased through one's efforts (an incremental theory). (p.198)

Implicit theories of intelligence influence the motivational goal that we feel driven to pursue. Entity theorists actively seek performance goals; in their framework, a task's outcome measures their limited intellectual capacity. Incremental theorists value effort as a conduit for success; therefore, they embrace learning goals and are motivated by mastering new things. Notably, implicit theories of intelligence can be domain-specific (Yeager & Dweck, 2012) and can operate in tension with the generally held theory.

As we navigate cognitive challenges, we continuously seek confirmation of our beliefs about intelligence, an endeavor referred to as "theory protection" (Plaks et al., 2005). As a result, the receipt of negative feedback (or stereotype disconfirming information) will cause both types of theorists to exhibit defensive processing, with more observed on the part of the entity theorists (Plaks et al., 2001). Defensive processing can impact our receptiveness to retain new information. For instance, following negative feedback on tests of general knowledge, Mangels et al. (2006) found that students' beliefs and reactions to failure influenced their learning success by manipulating their attention and conceptual processing, two functions that serve to either inhibit or increase gains in knowledge. Two examples of defensive processing are defensive inattention (a form of passive defense involving partial encoding of, or selective attention to, challenging information) and intensified scrutiny (a form of active defense involving discounting or debunking challenging information). When defensive inattention is not possible, intensified scrutiny may be employed (Eagly et al., 1999; Eagly et al., 2000; Plaks et al., 2005).

Our perception of negative feedback and its role in confirming or disconfirming our implicit theories of intelligence varies. To the entity theorist, negative feedback equates to failure in intellectual ability. Given a high grade, the entity theorist will continue to receive high grades; however, given a low grade, they will continue to receive low grades for their poor performance is a testament to their low, fixed intelligence that cannot be improved (Grant & Dweck, 2003). This response to failure is known as the helpless pattern and is characterized by the feeling that failure is out of one's control. As a result, entity theorists make ability attributions (e.g., "I'm not smart enough.") and are more susceptible to loss of self-worth (Grant & Dweck, 2003). Additionally, there can be a normative comparison element to performance goals (i.e., a desire to outperform others), which may lead to a reluctance to perceive one's performance as a failure in the first place (Grant & Dweck, 2003). In this regard, entity theorists may engage in intensified scrutiny, such as devaluing the problem, to preserve their perceived rank.

Incremental theorists exhibit a healthier response to failure as it poses no threat to their intellectual capacity. After a poor performance, the incremental theorist will make effort attributions (e.g., "I need to study more.") and will likely persist to the point of improvement (Grant & Dweck, 2003). This response is coined the mastery-oriented pattern and is characterized by linking failure to modifiable factors, such as lack of effort (Diener & Dweck, 1980). As a result, they will seek positive interpretations and growth (Diener & Dweck, 1980; Farrell & Dweck, 1985; Grant & Dweck, 2003; Mangels et al., 2006), which will ultimately lead to more significant gains in knowledge (Grant & Dweck, 2003).

Unsurprisingly, entity theorists' maladaptive tendencies can affect self-esteem and, in the case of mathematics, lead to mathematics anxiety. These learners are more likely to equate genius with low effort, an attribution which encourages them to value speed—with respect to recall of facts, the time it takes to solve a problem, and the general brevity of all mathematics solutions—over effort. Unfortunately, the role of speed in mathematics is misrepresented in popular culture, much to the detriment of mathematical learning: when we equate skill with speed and value fast recall over deep conceptual understanding, mathematics anxiety increases, and creative inquiry declines (Boaler & Zoido, 2016).

### Mathematical Discourse and Mathematical Identities

Several other misconceptions regarding mathematics are likewise promulgated by popular cultures, such as the various tropes that dominate our mental schemas regarding those characteristics that define a mathematician: the eccentric Einstein-like older man; the young, tortured genius; and the genetically different savant (Barba, 2018). Additionally, there is a "white male myth" regarding an innate proclivity for mathematics that permeates Western culture (Stinson, 2013). Not only does this myth exacerbate stereotype threat (e.g., race, gender), but it has been shown to impact the mathematics achievement of marginalized groups (Spencer et al., 1999; Steele, 1997; Steele & Aronson, 1995). These preconceived and developing notions that we have regarding mathematics and mathematicians shape our attitudes and preferences towards mathematics, two factors, of many, that contribute to one's mathematical identity.

Mathematical identity "can be broadly defined as participative, narrative, discursive, psychoanalytic or performative" (Darragh, 2016, p. 24). Theorists dispute its classification as conscious versus subconscious, independent versus interdependent, or an action versus an acquisition. Nonetheless, mathematical identity likely encompasses each of these attributes to some degree. In her examination of identity in mathematics education research, Darragh (2016) describes it as an adjustable lens through which a magnification reveals interactions on the individual scale and zooming out reveals interactions in a socio-political context. Then, she writes:

> We can look at the big picture, that is, at issues of mathematics learning in general. We can look at the experiences of specific groups of people and issues of equity. Or we can look at the individual level and try to understand learners' relationships with mathematics. (p. 20)

Regardless of the scale, social interactions are a critical element of mathematical identity. Thus, mathematical identities are developed and enforced via mathematics socialization through exchanges within "communities of practice" (Wenger, 1998) or "figured worlds" (Boaler & Greeno, 2000; Holland et al., 1998). Martin (2012) describes mathematics socialization as referring to "the experiences that individuals and groups have within a variety of mathematical contexts … that legitimize or inhibit meaningful participation in mathematics" (p.57). Arguably, social media has emerged as a source of mathematical socialization through which (non)mathematical identities are fostered. According to Epstein et al. (2010), young people use the mathematical discourse circulated in popular culture to negotiate their own identity making. Therefore, discourse is not only an integral contributor but also a conduit for identity formation. Every occasion for communication enables participants to construct and negotiate their self-image and social position (Davies & Harré, 2001; Waring, 2018). This negotiation is a perpetual process: mathematical identities are the byproducts of constant, and often subconscious, adjustments made from exposure to various narratives such as racial, gender, cultural, historical, or political.

Furthermore, mathematical identity is revealed in discourse through the negotiation of positional actions. Positioning is the reciprocal and dynamic process through which roles are actively established, altered, and reestablished for those engaged in the interaction (Davies & Harré, 1990). According to Davies and Harré (2001), "Positions are identified in part by extracting the autobiographical aspects of a conversation … to find out how each conversant conceives of themselves and the other participants by seeing what position they take up" (p. 264). Furthermore, "an explicit positioning of self naturally involves an implicit positioning of other" and vice versa (Minow, 2012, p. 98). Therefore, mathematical identity can be interpreted as the "social positioning of self and other" in mathematics discourse (Bucholtz & Hall, 2005, p. 586). Finally, the relative anonymity of social media emboldens users who feel immune to repercussions; as a result, their discourse can devolve into audacious criticism of others. Consequently, positioning acts are more conspicuous and intensify as social interactions expand from one-to-one to one-to-millions.

### Implicit Theories of Intelligence, Identities, and Positive Outcomes

According to Jetten et al. (2011), social interactions and identity can impact mental and physical health in a profound way. Extant studies have shown mathematical identity to be fundamental to the development of attitude, disposition, emotional well-being, and a general sense of self (Bishop, 2012). Additionally, mathematical identities are indicators of mathematical performance, persistence, and success (Cribbs et al., 2015). Implicit theories of intelligence have likewise been shown to be fundamental to academic success and linked with social interaction (e.g., adult feedback practices) (Blackwell et al., 2007; Plaks &

Stecher, 2007). For instance, incremental theorists receive higher grades, are reported to enjoy and value academics more, have increased motivation, choose more positive, effort-based responses to failure, are more resilient, demonstrate greater confidence, and experience greater overall gains than entity theorists (Aronson et al., 2002; Blackwell et al., 2007; Boaler, 2016; Dweck, 2016; Good et al., 2003).

However, studies have yet to show how social media interactions, primarily through written discourse, relate to implicit theories of intelligence and mathematical identities. Characterized by controversy, social media discourse surrounding mathematics problems is often a mélange of uninhibited reactions. Further, social media enables interaction among larger and more diverse groups of people. Therefore, it is important to view this particular form of discourse through a critical lens to determine the role it has in developing mathematical mindsets and identities, and its effect on positive outcomes, such as mathematics success.

## Purpose of the Study

The purpose of this study was to examine the discourse in the comments section of social media posts regarding a mathematics problem and analyze the underlying narratives which reveal the mathematical mindset and mathematical identity of each user.

## Method

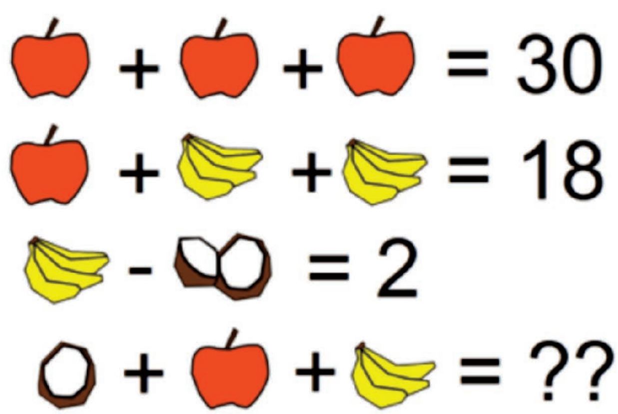
The current study focused on the discourse in the comments section of the same mathematics problem posted twice to YouTube in February 2016 (Figure 1). The mathematics problem was described as "simple-looking" and advertised as both an emoji mathematics problem and an algebra fruit puzzle. Both YouTube videos explained the controversy surrounding the problem, in particular, that it was first posted to Facebook, where it confused over two million people. Notably, the answer to the problem was given at the end of each video.

The comments of 1,046 YouTube users were examined (107 from the first video, and 939 from the second video). All comments were retrieved by the researcher by visiting each YouTube page and scrolling down until there were no remaining posts. Necessarily, this process was conducted over the same time period so that the posts appeared in the same order and could be tracked. Only original posts were studied; replies were only considered if the author of the original post engaged in discourse with other users. It was not possible to obtain any demographic information regarding each YouTube user.

The research followed a qualitative approach (Creswell, 2015) characterized by finding meaning through the subjective interpretation of participants' discourse. The phenomenon to be studied was the indicative nature of discourse to reveal a mathematical mindset and mathe-

**Figure 1**

*Viral Mathematics Problem*




**Note.** Talwalkar, P. (2016). Viral Facebook math problem stumping the internet. (https://mindyourdecisions.com/blog/2016/02/18/viral-facebook-math-problem-stumping-the-internet-answer-to-coconut-plus-apple-plus-banana/)

Mind Your Decisions

Presh Talwalkar

matical identity. The aim of the qualitative analysis was not to determine the number of YouTube users with the right answer but, rather, to investigate their discourse to identify the mathematical mindset and mathematical identity of each user.

Qualitative analysis began with coding strategies derived from Grounded Theory (Glaser & Strauss, 1967). Open coding (Boeije, 2010) was done mostly at the beginning of the data analysis. During this process, the researcher began to divide the posted comments into groups to form preliminary categories. The enumerated characterizations of these codes were then augmented during the axial coding process (Boeije, 2010) to boost the efficiency of the existing codes. Comments were only coded for one theme; however, if a user engaged in more than one comment, the username was tracked, and, in some instances, the initial code was changed.

In determining the quality of comments, underlying themes emerged, such as an apparent eagerness to boast about their intelligence, diminish the credibility of the mathematics problem, admit their faulty logic, denigrate their self-esteem, or voluntarily explain the solution for other users. To that effect, six codes were formed (Table 1).

After the initial open and axial coding process, selective coding (Boeije, 2010) was implemented in conjunction with Discourse Analysis (Waring, 2018) to determine the mathematical mindset and mathematical identity of each user (Table 2). Notably, mindsets can vary by subject (Yeager & Dweck, 2012) and operate in tension with the general mindset. Thus, the YouTube users in the present study were identified as having a mathematics specific mindset rather than a general mindset, as one cannot assume their general implicit theory of intelligence through the scope of a mathematical lens. Mathematical mindset was determined by examining written language indicators relative to attributions, motivational goals, response to failure, defensive processing, and normative comparisons. In line with an entity theory in mathematics, the fixed mathematical mindset is linked with ability attributions, performance goals, the helpless response to failure, passive and active defensive processing, and normative comparisons. Thus, to identify a social media user as having a fixed mathematical mindset, the researcher looked for indicators in their discourse that suggested the users (1) viewed efficacy as a measure of intelligence, (2) emphasized speed over effort, and (3) criticized the problem, such as devaluing or debunking it, to preserve their perceived rank.

In contrast, the growth mathematical mindset, aligned with an incremental theory in mathematics, is associated with effort attributions, learning goals, and the mastery-oriented response to failure. To identify a social media user as having a growth mathematical mindset, the researcher looked for indicators in their written discourse suggesting they (1) viewed efficacy as distinct from intellectual capacity, (2) sought positive interpretations of their failure, (3) valued effort over speed, and (4) were disinterested in their perceived rank.

Finally, the social interaction on the YouTube page allowed each participant an opportunity to reveal their mathematical identity via the self-image they wished to convey to their audience (Markus & Warf, 1987). The mathematical identity examined was interactional (Waring, 2018); therefore, mathematical identity was determined by the positioning acts (Davies & Harré, 1990) evident in the written discourse of each user. The type of communication studied was one-sided; thus, only first-order positional actions were considered. Six positions emerged from the analysis of discourse: (1) a position of superiority; (2) a position of authority/power; (3) a position of spectator; (4) a position of inferiority; (5) a

**Table 1**

*Coded Comments and their Characterizations Determined During the Open and Axial Coding Process*

| Coded Comment | Characterization |
|---|---|
| This is easy | • Emphasis on the short amount of time it took to solve the problem<br>• Emphasis on age<br>• Boasts about own intellectual ability<br>• Disparages people who get the problem wrong |
| This is not fair | • Disagrees with the presented solution<br>• Devalues the problem |
| I was wrong | • Willingly admitted they were wrong |
| I am not smart | • Denigrates self for getting the wrong answer |
| Let me explain my reasoning | • Provides instruction for other people in a non-disparaging way |
| Other | • Comments that did not resemble other categories and could not be consolidated into a category of their own<br>• Most often, single number answers to the mathematics problem |

Table 2

*Mathematical Mindsets and Identities Determined During the Selective Coding Process*

| Coded Comment | Characterization |
|---|---|
| **Mathematical Mindset:** Determined by examining written language indicators relative to attributions, motivational goals, response to failure, defensive processing, and normative comparisons | **Fixed:**<br>• Viewed efficacy as a measure of intelligence<br>• Emphasized speed over effort<br>• Scrutinized the problem, such as devaluing or debunking it, to preserve their perceived rank<br>**Growth:**<br>• Viewed efficacy as distinct from intellectual capacity<br>• Sought positive interpretations of their failure<br>• Valued effort over speed<br>• Were disinterested in their perceived rank |
| **Mathematical Identity:** Determined by written language indicators relative to positioning acts | **Position of Superiority:**<br>• Asserted their elevated proficiency in mathematics and desired to maintain their high standing<br>**Position of Authority/Power:**<br>• Asserted their superior proficiency in mathematics while simultaneously executing their authority to exert control over the narrative<br>**Position of Spectator:**<br>• Neutral bystanders to a mathematical debate<br>**Position of Inferiority:**<br>• Asserted their low normative comparison to others and desired to maintain it<br>**Position of Instructor/Solidarity:**<br>• Exhibited both an intent to encourage learning in other users and also solidarity in their understanding of how others had failed<br>**Position of Relative Indifference:**<br>• Disinterested in engaging further in discourse |

position of instructor/solidarity; and (6) a position of relative indifference. The emergent themes of positional actions found in the present study were like those described in Bishop's (2012) study of mathematical identities in the classroom.

## Analysis

### This Is Easy

This discourse exhibited an investment in performance with a focus on speed and age.  Speed was emphasized by users explicitly writing their time or using words to delineate their efficiency: "I found out the second I saw it." A link between age and mental prowess was emphasized by users indicating their grade level in school or writing comments such as "Got it right on the first try, and I'm 11."

Additionally, these users were eager to boast about their intellectual ability, writing comments such as, "Honestly, that was easy. I took the gifted test, and things like that were all over the place," and "It's really very simple for me to solve math problems." They also disparaged others who either got the problem wrong or were too baffled to find a solution. For example:

| |
|---|
| "Honey, I did this in year 1." |
| "2 million people are unable to answer the question. What dummies they are! Isn't it so simple?" |
| "This is a toddler's math problem." |
| "The sad thing is, someone thought it was 7." |

One user even denigrated the person that posted the problem: "You just need basic arithmetic to solve it, the person who posted it must be uneducated."

Ultimately, these users were identified as having a fixed mathematical mindset: they valued speed over effort, equated efficacy to intelligence, and cared about their normative comparison. Furthermore, their discourse was indicative of a mathematical identity dependent on their position of superiority: it was evident that they desired to assert their elevated proficiency in mathematics and maintain their high standing. Notably, their role in the narrative assigned an inferior status to the other users.

### This Is Not Fair

These users disagreed with the presented solution and devalued the problem. The discourse from these comments revealed defensive processing that impacted the users' ability to process new information, namely, that the problem was not algebraic. They were guarded, oversensitive, and contentious in their inability to accept failure. They engaged in intensified scrutiny to debunk or devalue the mathematics problem itself. Some were polite (e.g., "I'm afraid that you have a mistake in there") while others were blunt (e.g., "You are wrong"). Some went so far as to justify their "non-agreement":

> "I disagree since there is no '+' between the individual bananas and coconut halves adding them is not mathematically correct, you should multiply them instead of adding them, giving a final answer of 14.24264069."

> "At 2:06 you call the picture difference half a coconut ... but by the PICTURE they are not equal sizes … so we are splitting hairs in non-agreement. So depending on how you interpret the 'pictures' will adjust your answer. It boils down to doing the simple algebra properly and consistently. If you decide to be picture accurate though then you should consider using 2/3 for the last coconut picture yes?"

Whereas others exposed it as a popularity-generating scam: "These are designed to purposefully trick people to argue the answer, and create comments to buff popularity." Some even accused it as being a mostly observational problem (e.g., "1% maths and 99% observation"), denouncing it as a trick (e.g., "Fun vid but I lost interest when the 'trick' part came up") or an optical illusion intent on "pure deception." They even scrutinized the quality of the drawings:

> "It was clearly drawn poorly on purpose to cause problems."

> "That coconut looks more like 2/3 than 1/2."

> "This is IKEA's view on math problems making something really simple more difficult just because they [want] to draw pretty pictures."

There was also an abundance of sarcasm, "Maybe you need to count each pixel of the drawn icons separately," and insolence, "This is why we use letter variables instead of pictograph variables." Additionally, many of these comments were aggressive in their delivery, using expletives or all capital letters. Finally, some users employed more complex vocabulary and mathematics to assert their dominance over the correctness of the solution:

> "There's only ONE APPLE in the image representing a value of TEN. Thereby you cannot clearly establish a consistent rule that the images represent real rational numbers that can simply be counted by observing the image, only that there is a specific value as defined by a specific image. Inconsistent rules of variable declaration yields a [expletive] math problem."

Ultimately, these users were identified as having a fixed mathematical mindset: they engaged in defensive discourse and demonstrated a maladaptive response to failure. Furthermore, their discourse was indicative of a mathematical identity dependent on a position of authority. Their comments enforced their position of power by asserting their superior proficiency in mathematics while simultaneously executing their authority to control the narrative. They governed over the solution to the problem in an endeavor to subjugate those who disagreed with them.

### I Was Wrong

These users were willing to admit that they, and not the problem, were wrong. Most pointed out the component of the problem they failed to grasp, namely that the quantity of fruit was different:

> "Oh wow, never realized that the amount of fruit varied."

> "I noticed the coconut twist but didn't notice that there was one banana."

Some enjoyed being wrong: "Totally got me. That was fun!" While others were appreciative: "Yeah, I thought the answer was 16 too. I saw this puzzle on a social network, but because it was so easy, I didn't even look at the solution. Now I see things I never noticed before."

Ultimately, these users were identified as having a growth mathematical mindset: they were confident enough in their mathematical ability to the extent that this single mathematical problem did not threaten their intelligence, behavior characteristic of the mastery-oriented response to failure. Furthermore, their discourse was indicative of a mathematical identity comfortable with the position of spectator; these users positioned themselves as bystanders to a grand mathematical debate. They played a neutral role in the narrative, neither asserting themselves as above nor below another user.

### I Am Not Smart

These users not only willingly admitted they were wrong but were self-denigrating in the process, depicting a clear loss of self-worth. In addition to lamenting their low intellectual ability, "Why am I so dumb?", they showcased their arithmetic errors in a disparaging tone: "lol I thought 18-10 was 9, so smart." One particular user volunteered two different answers in two different comments and surrendered in a third comment: "Well screw that." Another user went so far as to explain their reasoning in a relatively lengthy post, only to conclude with "I'm gonna get this wrong anyway."

Ultimately, these users were identified as having a fixed mathematical mindset: their self-identification as "dumb" suggests their subscription to the belief that performance is indicative of intelligence. Furthermore, their discourse was emblematic of a mathematical identity dependent on a position of inferiority. These users lacked faith in their mathematical skills, demonstrating discomfort in mathematical socialization and suggesting an abundance of non-mathematical identities. Not only did they assert their low normative comparison to others, but they desired to maintain it, thereby enforcing the superior position of others.

### Let Me Explain My Reasoning

These users offered insight to the problem in a markedly non-disparaging way:

> "It is a really simple problem; however, most people did not know that the final answer had to consider the change in quantity of each fruit."

> "The problem people have is they keep changing it to variables. It is pictures of fruit, not letters representing numbers."

Some even explained the solution using real-life scenarios:

> "You go to a shop and see packs of bananas at a discount. 1 pack = 1 euro. You notice that a few of the packs contain just 3 bananas while most packs contain 4. Would you buy the 3-pack??? My point is: we can never afford to dismiss the importance of attention to detail."

They even demonstrated positive growth interpretations from past failures: "I have learned to look a little closer in these things."

Ultimately, these users were identified as having a growth mathematical mindset: they made effort attributions in their constructive criticism of other users' approach to solve the problem. Furthermore, their discourse indicated a mathematical identity emblematic of instructor and solidarity with others; they exhibited both an intent to encourage learning in other users and solidarity in their understanding of how others had failed. Distinct from a position of authority in which the desire was to exert power and control over others, these users expressed a desire to rectify others' mistakes.

### Other

All other comments were categorized as "other." Most of these comments consisted of single-number solutions to the mathematics problem. These users demonstrated a clear lack of desire to engage in discourse with others. Due to the ambiguity of motive and lack of sufficient verbiage, it is not possible to determine the mathematical mindset of these users. With that being said, their disinterest is indicative of a mathematical identity dependent on a position of relative indifference.

## Results

Of the 1046 comments, the following mathematical identities were revealed through discourse: 170 (16.3%) wrote from a position of superiority; 135 (12.9%) from a position of authority/power; 41 (3.9%) from a position of spectator; 7 (0.7%) from a position of inferiority; 34 (3.3%) from a position of instructor/solidarity; and 659 (63%) from a position of indifference. Additionally, 312 (30%) used discourse suggestive of the fixed mathematical mindset, whereas only 75 (7.2%) of comments were indicative of the growth mathematical mindset.

Table 3
*Results*

| Coded Comment | Associated Mathematical Mindset | Associated Mathematical Identity | Percentage |
|---|---|---|---|
| This is easy | Fixed | Position of Superiority | 16.3 |
| This is not fair | Fixed | Position of Authority/Power | 12.9 |
| I was wrong | Growth | Position of Spectator | 3.9 |
| I am not smart | Fixed | Position of Inferiority | 0.7 |
| Let me explain my reasoning | Growth | Position of Instructor/Solidarity | 3.3 |
| Other | NA | Position of Relative Indifference | 63.0 |

## Discussion

Social media discourse is presently understudied. Extant studies have demonstrated the importance of fostering productive mathematical mindsets and mathematical identities and the integral role that discourse (e.g., classroom, parent to child) plays in their development and progression; however, social media discourse is vastly different from most conventional forms. First, mathematics problems on social media generate controversy. Their portrayal as puzzles only geniuses can solve naturally incites competition. Second, the absence of an answer, or even a collective dismissal of the perceived answer, leads to heated disputes. In fact, authority on social media is sometimes denigrated as opinion. Third, the unfiltered discourse surrounding these posts encourages an unbridled comment culture exemplified by the uninhibited and audacious criticism of others. Fourth, social media discourse is primarily written, limiting users to modern written language indicators of expression. Finally, social media generates a larger, more diverse community than that typically studied. It allows for a unique forum of mathematical discourse that intensifies as the post grows in popularity. Therefore, it is important to examine how social media discourse contributes to mathematical mindsets and mathematical identities.

This study found discourse in social media to indicate both mathematical mindset and mathematical identity; furthermore, mathematical mindset and mathematical identity were linked. The fixed mathematical mindset corresponded to mathematical identities that positioned the user as superior, inferior, or authoritative. This is unsurprising, as the interest each of these users had in ranking their mathematical ability and asserting their relative comparison to others is typical of ability attributions and performance goals. In contrast, the growth mathematical mindset corresponded to those mathematical identities that positioned the user as spectator or instructor/solidarity. Notably, these users made effort attributions and showed complete disinterest in their comparison to others, suggesting their mathematical identities were more robust because they were unthreatened by performance indicators.

Social media can inspire confidence and engender positive change; however, it is necessary to transform harmful notions of efficacy in mathematics and false narratives of what it means to be a mathematician. Arguably, those users that engaged in the most detrimental discourse were those whose intellectual capacity and normative comparison were threatened by their failure. Positional acts are reciprocal; therefore, these users played supportive roles in developing the mathematical identities of others. It is only by understanding the integral role that mathematics socialization in various arenas has in developing mathematical mindsets and mathematical identities that we can enhance mathematical learning and encourage mathematical success.

Future studies should determine further the extent to which mathematical mindsets and mathematical identities are related through positional actions in social media discourse and if the same positioning acts are linked consistently with the same mathematical mindsets. Future studies should also explore how mathematical mindsets and mathematical identities are expressed through positional actions of discourse on social media platforms other than YouTube. Are certain mindsets and identities more prevalent on certain sites? Does the language used by users change as they switch between social media applications? How does student discourse in the classroom relate to student discourse on the Internet, and which indicates their true mathematical mindset and identity?

Additionally, it is important to understand that mindset and identity are multidimensional and should be examined on a spectrum. In fact, the development of these two constructs is continuous and dynamic. Therefore, interventions, such as those that promote healthier mindsets and identities, can embolden learners to reach higher levels of mathematical efficacy. With this knowledge, educators can better equip themselves, and their students, with those speech patterns that promote the mathematical growth mindset and positive mathematical identities. Furthermore, they should better prepare their students to be resilient when engaging in mathematics discourse on social media. Finally, they should be more cognizant of the discourse being used outside the classroom and its effects inside the classroom.

A possible limitation to this study is the lack of detailed analysis of the "other" category, which made up for 63% of the comments. It is difficult to ascertain the motive behind single-word discourse. Perhaps these users were confident in their mathematical abilities to solve the problem with no elaboration. Or maybe they skipped to the end of the video and copied the answer, thereby posting their solution to convince others that they solved the problem. Alternatively, maybe they simply did not care, or maybe they cared just enough to let people know they were "smart." Regardless, their desire to post yet not fully contribute to the discourse is similar to the mathematical identity emblematic of spectator and should be studied further.

Another possible limitation of this study is the subjective interpretation of the coding process. The researcher ensured the validity of the coding process via close reference to the tables of characterizations. However, without the context of tone from spoken language, nonverbal cues, or further questioning by the researcher, it is possible that comments could have been attributed to different coded themes. Future studies should be conducted which include inter-rater reliability. Additionally, in cases where comments may align with more than one coded theme, future studies should incorporate coding comments to more than one theme.

Social media use is on the rise, and its growth has sparked an evolution of, and dependence on, written discourse. Unfiltered and widely disseminated, it is important to increase our understanding of the impact of social media posts. Already, positive and negative health outcomes have been reported from social media use. Thus, it is increasingly crucial that educators recognize the effect that social media interactions have on their students. Ultimately, this unique form of discourse can be used as a conduit for mathematics success through its relation to mathematical mindsets and mathematical identities.

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# Mathematics Assessment at the Postsecondary Level: Three Alternative Forms of Assessment

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**ABSTRACT** Postsecondary mathematics classrooms continue to become more diverse, which implores educators to continue to incorporate more diverse modes of assessment to support all learners. However, researchers have found that mathematics assessment at the postsecondary level has remained mostly stagnant, with traditional, closed-book exams dominating the field. In this paper, we present three alternative forms of mathematics assessments for teachers at the postsecondary level. We incorporate researched-based tools for implementation and grading online, oral, and project-based assessments in mathematics. Potential limitations of each type of assessment are also addressed.

**KEYWORDS** *mathematics assessment, online assessment, oral assessment, project-based assessment, postsecondary mathematics*

Mathematics assessment in the postsecondary classroom gathers information about student content knowledge and their mathematics learning, provides students with feedback on their learning, and helps teachers reflect and improve their teaching practices (Suurtamm et al., 2016). Strategies for mathematics assessments have remained somewhat stagnant at the postsecondary level, with traditional, closed-book exams dominating the field (Iannone & Simpson, 2011, 2015). With an increasingly diverse student body entering postsecondary mathematics courses, teachers have an obligation to accommodate a variety of academic and professional needs. Incorporating alternative forms of assessment into a mathematics course can help teachers create a well-rounded evaluation of students' knowledge and skills. This paper includes resources and strategies for three types of alternative mathematics assessments: (1) online, (2) oral, and (3) project-based. Each section will define these alternative assessments in the context of a postsecondary mathematics classroom, provide specific research-based resources for implementation and grading, and address concerns about potential limitations.

## Online Assessments

Assessments must measure student comprehension of learning objectives, provide students an opportunity to self-assess, and cultivate a feedback dialogue between teachers and students (Robles & Braathen, 2002). Online mathematics assessments fulfill these requirements and most aspects of traditional paper-and-pencil assessments. Advances in technology allow for multiple choice, true/false, matching, fill in the blank, free response, and many other question types to be uploaded into a digital format and instantly graded through the use of algorithms (Herbet et al., 2019; Pelkola et al., 2018). Further, many online assessment resources provide immediate feedback to student responses (Joglar et al., 2010), and some have the capability to generate personalized questions based on students' progress (Herbert et al., 2019). Online platforms for course management, learning, and teaching, known as *learning management programs,* are also equipped for teachers to build their online assessments and then import students' data into their grade books. Most online assessments can support mathematics editing codes such as LaTeX and allow the teacher to insert interactive multimedia files from sources like GeoGebra and Desmos (Joglar et al., 2010). Several online assessment platforms allow teachers to share the resources they have created, thereby constructing digital banks of mathematics assessment questions (Gleason, 2012; Joglar et al., 2010; Pelkola et al., 2018).

### Online Assessment Resources

Many resources are available for instructors to create online mathematics assessments. Bolster Academy, a standalone program, specializes in open-ended questions for students taking advanced mathematics courses at the postsecondary level with automated feedback and shared question banks (2020). Maple T.A., or Möbius, specializes in Science, Technology, Engineering, and Mathematics (STEM) courses, offers a variety of question types, automated feedback, and shared questions banks (Maplesoft, 2020). Maple T.A. is compatible with most learning management programs and also works as a standalone program. Both Bolster Academy and Maple T.A. are online programs that the teacher, students, or institution must pay for a subscription; however, a number of open-source, freely distributed programs that teachers can contribute to and modify to suit their needs are available. Lumen OHM, also known as MyOpenMath, provides teachers with question banks in which they can share and use questions designed for specific mathematics content areas (IMathAS, 2020). MyOpenMath also automatically grades multiple choice, numerical, and graphing solutions, providing students with instant feedback (IMathAS, 2020). WebWork, supported by the Mathematical Association of America and the National Science Foundation, focuses on formative assessment in the form of homework with automated feedback and shared question banks (The Mathematical Association of America, 2020). Teachers using Google Classroom may also want to take advantage of Google Forms. While not math specific, Google Forms allows teachers to manually enter feedback and automatically generate a statistical report of students' responses. We created a more detailed description of these online assessment resources and their capabilities that can be found in Appendix A.

Teachers can also access online resources available on their school's learning management programs to create assessments. Some of the most commonly used learning management programs such as Blackboard, Brightspace, Canvas, Moodle, and Sakai have a variety of question types that provide both automated and manually entered feedback. Each of these systems generates statistical reports of students' performance incorporated into the system's grade book. It is important to note that these programs are not explicitly designed for mathematics assessments; instead, they are designed to be comprehensive enough to permit use in a wide range of disciplines. Advanced mathematics teachers and instructors may feel limited by the available built-in math editors, which often do not accommodate such things as scientific notation, graphing, or diagrams. Teachers could alternatively ask students to upload files or pictures of their work to be manually graded.

### Implementing and Grading Online Assessments

Initially, implementing an online mathematics assessment will create some challenges for instructors, as we will address in the next section. Teachers must become familiar with the program's format and design while building a database of questions (Joglar et al., 2010). However, paid programs like Maple T.A. or Bolster Academy and open-source programs like MyOpenMath and WebWork have mathematics question banks which teachers can access for formatting examples or for incorporating directly into their assessments. Furthermore, online mathematics programs like GeoGebra and Desmos have open-source, pre-built models that can be embedded into online assessment resources, thereby alleviating the technological strain on teachers in building their own models. We recommend that teachers search GeoGebra and Desmos for specific diagrams, graphs, or models, and then copy the embedding code and insert it into the assessment system within their learning management program.

As noted above, most of the online assessment programs are able to incorporate most mathematics questions, excluding *Free Response/Essay* and *File Upload,* to be automatically graded and provide instant student feedback. For example, in the learning management program Canvas, teachers can select the point value of each question and customize the type of feedback to include simple correct/incorrect hints, or further directions. Blackboard, Brightspace, Canvas, Moodle, and Sakai have built-in grade books which will automatically fill in students' data and create statistical reports for teacher evaluation. Most online assessment programs also collect student data; Lumen OHM, for instance, will calculate students' scores for assessments and course averages that can be downloaded into an Excel spreadsheet.

## Potential Limitations to Online Assessments

### *Accessibility*

There are limitations to the use of online mathematics assessments as an alternative to traditional paper-and-pencil assessment. Student and teacher accessibility to computers and reliable Internet will pose the most formidable of challenges (Greenhow, 2015; Herbert et al., 2019), with the potential to unintentionally discriminate against those who do not have the resources to access the online assessments. As the COVID-19 Pandemic has forced schools around the world to move classes online, this issue has been exacerbated, and teachers have found that many students lack the resources necessary to participate (Reilly, 2020). We recommend that teachers gather information about students' accessibility privately before implementing any mandatory online mathematics assessment. Teachers, alternatively, may choose to implement mandatory assessments in school computer labs to ensure accessibility for all students.

### *Technological Skills*

Initially, the technological skills of both the teachers and students may pose a potential limitation. Teachers and students need time to become comfortable using these online mathematics assessment tools and familiarize themselves with the selected programs before using the assessment in a high-stakes environment (Herbert et al., 2019). Depending on the audience, it may be useful to provide in-service training to help teachers become comfortable with online mathematics assessment programs (Joglar, 2010).To facilitate effective student use of programs, Greenhow (2015) suggests mock online assessments that allow students multiple attempts or an initial assessment in a classroom setting with the teacher present to address questions and issues in real time.

### *Academic Dishonesty*

Teachers' apprehension over cheating and academic dishonesty is often heightened when considering online mathematics assessment (Kennedy et al., 2000; Ladyshewsky, 2015). Online question and answer websites, such as Chegg.com, have enabled cheating, particularly on mathematics assessments inclined towards single solutions and numerical responses (Klein, 2020; Supiano, 2020). A number of ways to combat the risk of academic dishonesty for online mathematics assessments include: using the question-randomization option on assessment tools, lowering the stakes of the assessment, scheduling a time and setting a time limit for the assessment, or asking critical thinking questions instead of multiple choice or true/false (Ladyshewsky, 2015). Randomized questions that have time limits reduce students' ability to share and search for answers online (Ladyshewsky, 2015). Harmon and Lambrinos' (2008) study found fewer instances of cheating when a proctor supervised the assessment. While some schools may have the resources to hold online assessments in computer testing centers with proctors, this is not always an option, especially for students taking courses online. Online assessment integrity resources such as Honorlock, ProcturU, and Proctorio provide teachers with secure online proctoring software that monitors students virtually as they assess, provide identity verification, and professional review for signs of academic dishonesty. We recommend that teachers ask their Information Technology Department to see if this type of software is available from their online campus. It is important to note that there has been recent backlash to these integrity resources, with reports of students' feeling an invasion of privacy, increased anxiety while testing, and a plethora of technological issues (Patil & Bromwich, 2020). Teachers, alternatively, may consider having students sign an honor pledge or code, where they commit to honestly completing the assessment. Honor codes have been found to reduce cheating and support integrity on assessments of all types (Miller, 2020).

## Oral Assessments

Oral assessment in mathematics is not common in American classrooms, unlike countries such as Hungary, Italy, and the Czech Republic who commonly employ this assessment method in their higher education courses (Iannone & Simpson, 2012). Lee (1988) describes learning as "more than a paper and pencil activity" (p.12), and oral assessments provide a space for students to show off their "problem solving skills rather than quick answers" (Sayre, 2014, p.30). Oral assessments help develop students' communication and logical reasoning skills (Chasteen, 2018; Iannone & Simpson, 2011; Joughin, 2010). Moreover, oral assessment in mathematics can significantly reduce, if not eradicate, cheating, and plagiarism among students (Joughin, 1998, as cited in Iannone & Simpson, 2011).

Joughin (2010) categorizes oral assessments into three types: *presentation, application,* and *interrogation.* In the postsecondary mathematics setting, the two most prevalent forms of assessment from Joughin's (2010) model are *presentation* and *interrogation. Presentations* are characterized as an "in-class presentation on a prepared topic [or a] group project report to the class" (Joughin, 2010, p. 3). *Interrogations* are described as the process where a student is quizzed or interviewed by the instructor (Joughin, 2010); we will primarily be discussing oral assessment in mathematics as an *interrogation.*

### Setting the Stage for Oral Assessment in Mathematics

An instructor wishing to implement an oral assessment in mathematics needs to consider the classroom setting, which will dictate such an endeavor's plausibility. Researchers most commonly execute and study the effectiveness of oral assessments in small classroom settings (Iannone & Simpson, 2012; Odafe, 2006; Sayer, 2014), suggesting that a smaller classroom's intimacy cultivates a more suitable environment. Furthermore, the instructor needs to thoroughly plan for the assessment, asking themselves about the types of interactions and questions needed to properly assess students' mathematical knowledge, who the audience will be, and how to structure the assessment (Joughin, 2010). To reduce ambiguity and confusion for students, it is recommended to discuss the oral assessment format and expectations beforehand and provide written directions (Iannone & Simpson, 2012; Joughin, 2010). Practitioners suggest implementing practice oral quizzes and providing detailed one-on-one feedback to help familiarize students with the assessment process (Dumbaugh, 2020; Iannone & Simpson, 2012). Moreover, a comprehensive rubric of how students will be graded should be available prior to being assessed (Odafe, 2006), with the assessment's intentions clearly stated as a high or low stakes test (Iannone & Simpson, 2012).

### Implementation of Oral Assessments

In mathematics, oral assessments are generally administered in two ways: group oral assessment with three to four students (Odafe, 2006) or individualized oral assessment (Boedigheimer et al., 2015; Iannone & Simpson, 2012). These types of assessments are recommended to be held outside of regular class time and should not exceed more than 60 minutes (Chasteen, 2018; Iannone & Simpson, 2012; Odafe, 2006; Sayre, 2014). Instructors may also choose to employ teaching or course assistants to increase the efficiency of administering oral assessment; however, this is only recommended for use in practice or in a low-stakes environment (Chasteen, 2018). The assessment should feature "harder, more interesting problems than…a written exam" (Sayre, 2014, p. 32) and should be both thought-provoking and not invite one-word numerical answers.

#### *Group Oral Assessments*

In a group oral assessment setting, it is important to have students accustomed to collaborating on mathematics problems in groups (Chasteen, 2018; Odafe, 2006). We recommend that teachers create groups of three to four students. In Odafe's (2006) example, students were assessed with the same group they were assigned to during class time and were provided with space, such as a whiteboard, to express mathematical ideas in written format. Some researchers recommend that scripted questions be pulled randomly from a collection created by the instructor in advance (Chasteen, 2018; Odafe, 2006). Teachers should work to create a dialogue with students as they are being assessed, asking such questions such as "Why did you do that?", "Can you explain an alternate way of solving the problem?" and allowing group members to assist each other when necessary (Lee, 1988; Odafe, 2006). The amount of time allotted for each group will vary depending on the length and difficulty of the questions. For example, Odafe's (2006) study of group oral assessments in a College Algebra course allotted approximately 30 minutes for each group, allowing students to complete two to three problems. Chasteen's (2018) Calculus courses allotted for up to an hour for each group.

#### *Individual Oral Assessments*

Current research indicates that 10-30-minute interview sessions are sufficient to assess individual students' understanding of mathematics topics (Boedigheimer et al., 2015; Dumbaugh, 2020; Iannone & Simpson, 2012). In Iannone and Simpson's (2012) study, students were tested on two questions: one in theoretical and one in applied mathematics. The student was allowed to choose the first question from either category and respond. The tutor would then randomly select a question from the other category for the student to respond to (Iannone & Simpson, 2012). When posing theoretical mathematics questions, research suggests that keywords such as prove, explain, and draw can be used to elicit responses that demonstrate students' reasoning and understanding (Chasteen, 2018; Iannone & Simpson, 2012). Applied mathematics questions in an oral assessment should ask students to implement algorithms to solve problems in front of the instructor as they talk through their thought process (Iannone & Simpson, 2012). Sayre (2014) suggests that students should not perform tedious calculations of a problem; instead, the instructor should guide the student to focus on the content and reasoning behind the problem. Odafe (2006) further recommends that instructors incorporate ample wait time for student responses, and, at the conclusion of an oral assessment, address misconceptions or mistakes made.

### Grading Oral Assessments

Grading oral mathematics assessments requires several components to facilitate a fair and equitable process. Firstly, a clear rubric that rates students on their solution(s), the communicated ideas and application while problem solving orally, and the clarity of their explanations (Boedigheimer et al., 2015; Iannone & Simpson, 2012;). Students can also be rated on how far they can carry the question through, with or without varying levels of assistance (Odafe, 2006; Sayre, 2014). It is recommended that instructors, either video or audio record the sessions for review or in the case of an appeal (Iannone & Simpson, 2015; Joughin, 2010). Teachers are advised to administer grades only after notes, comments, and recordings have been reviewed (Odafe, 2006).

### Potential Limitations to Oral Assessments

#### *Assessor Bias*

One of the major concerns for oral assessment in mathematics is fairness and the mitigation of assessor bias (Iannone & Simpson, 2012; Joughin, 2010; Sayre, 2014). In addition to clearly communicated expectations, Sayre (2014) recommends postponing grading until all students have completed the oral examination to address grading fairness. While assessor bias, intentional or unintentional, may be somewhat unavoidable, the use of video or audio recording allows for "ways of subsequently monitoring the process and moderating for the marks" (Iannone & Simpson, 2012, p. 180). To minimize assessor bias, courses with multiple teachers may also randomly assign students to be assessed by instructors of other sections (Boedigheimer et al., 2015).

#### *Stress and Anxiety*

Another limitation found with oral assessment in mathematics is the high levels of stress or anxiety students can experience while testing (Iannone & Simpson, 2012; Joughin, 2010). Iannone (2020) found that student nervousness arises from two factors: (1) interacting with the instructor as they are taking their assessment, and (2) the unpredictability of the questions posed. To mitigate the first factor, Iannone (2020) found that consistent dialogue with students in the classroom made students more comfortable with the instructor. For the second factor, Odafe (2006) provides students with a pool of ten questions from which the oral exam will be composed one week prior to the examination. Alternatively, other research has found that oral assessments may benefit students who find themselves suffering from math anxiety associated with written examinations (Heath, 1994). Some students, such as those with dyslexia or vision impairments, may find expressing their thinking and understanding of mathematics concepts orally less stressful (Joughin, 2010).

#### *Time Consumption*

Time consumption for the administration of oral examinations can often seem daunting to practitioners (Joughin, 2010; Odafe, 2006). Large class sizes can create a barrier due to the time constraints for both teachers and students (Boedigheimer et al., 2015; Joughin, 2010), particularly if scheduling assessment outside of class time. Sayer (2014) recommends that instructors consider the class size before choosing oral assessments as an alternative, noting that she will not use oral exams in classes much larger than 25 students.

#### *Student Needs*

Lastly, it is important to acknowledge that oral assessment in mathematics may not be suitable for all learners. Some students may have hearing or speech impairments that would make oral examinations discriminatory (Joughin, 2010); others may not be experienced at explaining their reasoning or thinking on-the-spot (Odafe, 2006). Also, international students may not be proficient in the language that oral assessment is conducted, once again making the assessment unintentionally discriminatory. The instructor needs to consider these limitations and make adjustments accordingly.

## Project-Based Assessments

Project-based learning (PjBL), to demonstrate real-life applications, made its debut in the early 1900s in the United States (Barron et al., 1998). PjBL "is a comprehensive approach to classroom teaching and learning that is designed to engage students in the investigation of authentic problems" (Blumenfeld et al., 1991, p. 379). While project-based assessments (PjBA) are generally used in classrooms that teach with PjBL, they are versatile enough to be a part of any mathematics course. Helle et al. (2006) describe PjBA and PjBL as having five essential features: (1) the problem drives the investigation, (2) students construct a concrete product, (3) students are in control of their learning process, (4) the problems and solutions are contextual and challenging, and (5) students can represent their solutions in multiple formats.

The use of PjBA has been praised as a type of authentic assessment, that serves as a method of evaluation while continuing student learning through providing opportunities for "meaningful experiences … [and] … high-level thinking" (Fauziah, 2018, p. 1). Given the investigative nature of PjBA (Blumenfeld et al., 1991), they are excellent platforms for students to apply their mathematical content knowledge while exercising critical thinking skills in both mathematics and project design (Helle et al., 2006). PjBA in mathematics encourages students to engage in problem-solving, perform experiments, analyze data, or create presentations (Blumenfeld et al., 1991; Russell & Rowlett, 2019). PjBL ensures that students overcome misconceptions that could easily be overlooked in traditional learning environments (Helle et al., 2006). PjBA can also be a tool that integrates the fields of STEM, helping students to recognize and apply the relationships between disciplines (Han et al., 2016). Furthermore, traditional academic skills students acquire in typical undergraduate mathematics programs are not always employable. Researchers (Hibberd, 2005; Knight & Yorke; 2004, as cited in Russell & Rowlett, 2019) claim PjBL and PjBA can build on desirable professional skills (Hibberd, 2005), such as how to collaborate in a professional group setting while improving planning and organizational skills (Russell & Rowlett, 2019). Similar to oral assessments in mathematics, PjBA will require careful planning and clear communication of expectations to students.

### Types of Project-Based Assessments

The versatility of PjBA means it can take on many different forms, such as portfolio projects, academic papers, or presentations. Portfolios consist of a collection of high-quality student work throughout a course that highlights their mathematical explorations and abilities (Knoerr & McDonald, 1999). Portfolios provide students with an opportunity "to take an active role in their own assessment and progress toward completing course objectives" as well as to "present a full assessment of learning" (Burks, 2010, p. 455-457). In academic mathematical papers, students research a topic applicable to the content of the mathematics course and write about it in their own words (Keith, 1988). Crannell (1999) promotes academic papers as a way: "(1) to improve students' mathematical exposition; (2) to introduce new mathematics; (3) to strengthen understanding of previously encountered mathematics; and (4) to provide feedback from the student to the instructor" (p. 113). Lastly, presentations can be live or video recordings of students' work on a mathematical problem, explanation of a mathematics theory or concept, or application of mathematics to conduct an experiment.

### Implementation of Project-Based Assessments

When implementing PjBA in a mathematics course, there needs to be a transitional phase that gives students opportunities to learn about the processes of PjBL and engage in initial low-stakes assessments (Blum, 1999; Slough & Milam, 2013). Instructional strategies such as scaffolding each step in a PjBA, modeling with exemplars of different parts of a PjBA, and encouraging perseverance will help students to transition to this new style of assessment (Barron et al., 1998; Slough & Milam, 2013). Teachers should assume an advisory or facilitator role rather than an authoritarian role (Adderley et al., 1975, as cited in Helle et al., 2006). Moreover, Slough and Milam (2013) recommend that during the PjBA process, teachers ensure: (1) content is made accessible to students, (2) the thinking process is visible, "which includes visual elements to help the learner and using learner constructed visual elements to assess learning" (p.16), (3) students are encouraged to learn from each other, and (4) the PjBA is focused on "autonomy and lifelong learning" (p. 16). Students will need to become accustomed to working together, communicating mathematical ideas, giving and receiving feedback, and understanding how to create a product in a timely and organized fashion. As PjBA ordinarily occurs within a group setting, instructors are encouraged to have students select roles, draft the goals and ground rules, or even establish cooperation agreements (Capraro & Corlu, 2013; Morgan & Slough, 2013) (see *Personal Accountability*). As with the case with oral assessments, students should be familiar with how they will be assessed at the beginning of a PjBA, provided with rubrics, prompts, and checklists.

### Grading Project-Based Assessments

#### *Rubrics and Checklists*

In PjBA, "rubrics are an essential component … that serve different purposes for those who are involved in the assessment process both at the stage of the rubric's development and its utilization during the evaluation" (Capraro & Corlu, 2013, p.115). To promote deeper understanding and involvement of the assessment process, researchers (Capraro & Corlu, 2013; GAIMME, 2016) encourage teachers to allow students to be a part of the rubric design process and then use these rubrics as tools for self- and peer-assessment. Capraro and colleagues (2013) include rubric categories such as authenticity, academic rigor, exploration and independent research, use of technology, and application and demonstration of learning. Checklists can also serve as a scaffolding tool for students and teachers, providing guidance on formatting and important components for completing PjBA (GAIMME, 2016). Checklists should be catered to the type of PjBA and include such items as how to structure a speech or advice on time management (Doree, 2017).

#### *Personal Accountability*

In group PjBA, students will need to be assessed both individually and as a group. Capraro and Corlu (2013) suggest that teachers use peer assessments, student reflections, group contracts, or an additional individual assessment to help increase individual accountability and fairness. Peer assessments either direct students to use the predetermined rubric(s) to assess each other or employ a Likert survey of teammates' contributions. In PjBA, students prefer to "demonstrate what they know instead of being caught at what they don't know" (Kenschaft, 1999, p.133). Teachers can provide space to demonstrate this knowledge through student reflections in the form of a survey or essay to include an account of personal responsibility and contribution to the project, in addition to a self-assessment using an agreed-upon rubric. Group contracts can include items related to behavior and social interactions in the group, responsibilities for each member, and consequences for negative actions (Capraro & Corlu, 2013). The combination of these documents, with the rubric(s) or checklist(s), can then be used to create a weighted grading system that will evaluate the whole process of the PjBA.

### Potential Limitations to Project-Based Assessment

#### *Time and Class Size Constraints*

PjBA provides a versatile platform for mathematics assessment that builds on students' ability to problem solve and encourages student-centered learning throughout the process; there are a number of potential limitations. Most notable is the constraint of time; PjBA is a process that will take more than one class period to complete and often requires students to collaborate outside of class (Morgan et al., 2013). In addition to providing time for students to work on the project during class, more time will be required should teachers include presentations of projects as part of the assessment. Teachers should also consider the time they will need to grade PjBAs. It may require a more in-depth analysis of items submitted by the students; however, the use of comprehensive rubrics and clear expectations can simplify grading.

As with oral assessments, large class sizes pose a limitation to PjBA. It may be challenging for the teacher to guide and monitor students throughout the whole process. In such cases, it is recommended that the teacher use group-based PjBA and seek out "peer facilitators" that can help to supervise students (Özel, 2013).

#### *Plagiarism*

Unlike online and oral assessments in mathematics, PjBA is more susceptible to plagiarism (Johnson, 1983). Students may unintentionally plagiarize using ideas or content from the Internet without proper citation, or intentionally, by looking up solutions to problems. Unintentional plagiarism can primarily be mitigated by using explicit guidance on when and how to cite sources. Intentional plagiarism can be reduced by using original problems posed by the teacher or selected by the students.

#### *Lack of Prior Knowledge and Skills*

Lastly, PjBA may pose initial challenges to students, who may not have the prior knowledge or skills necessary to implement a PjBA all on their own (Capraro & Jones, 2013). Skills will differ from student to student, and teachers will need to consider this varying level of experience prior to engaging in a high-stakes assessment (Capraro & Corlu, 2013). Researchers suggest the use of scaffolding to include coaching students through a project or modeling the expected outcomes of the PjBA (Slough & Milman, 2013). The use of instructional strategies mentioned in the implementation section, along with time and experience, will temper these challenges with PjBL and PjBA.

## Conclusion

The use of alternative forms of assessment in postsecondary mathematics classrooms provides diverse student populations multiple opportunities to showcase their strengths and abilities. Online mathematics assessments increase accessibility to both formative and summative forms of assessment while providing reusable resources for teachers that are both academically challenging and efficient at data collection and analysis. Oral assessments provide an authentic portrayal of students' understanding and knowledge of mathematical content while building communication and problem-solving skills. PjBA facilitates high-level thinking of real-world problems that develop students' professional skills.

We found it important to mention that during the COVID-19 Pandemic, teachers may modify these forms of alternative mathematics assessments to suit the needs of students in online or hybrid classrooms. We recommend that teachers continue to follow the above research for implementing, grading, and addressing potential limitations of alternative mathematics assessments. Teachers may also want to consider increasing the use of low-stakes, formative, online assessments to gauge student understanding during distance learning using the resources noted in Appendix A. Oral mathematics assessments can be administered using a video-conferencing medium, where teachers provide a digital whiteboard for students. Most PjBAs can be submitted digitally, and teachers can use video-conferencing mediums to conduct live presentations. Teachers may also want to consider asking students to create pre-recorded presentations to submit as part of their PjBA; these can then be shared through a discussion board on a learning management program or played during class.

Although this is not an exhaustive list of alternative forms of assessment, this paper was designed with the intention that postsecondary teachers and instructors will implement these or other alternative forms of assessment in their classrooms. We hope that these resources and ideas will help teachers become more flexible and innovative in their assessment strategies in a variety of postsecondary classroom settings. Teachers are encouraged to use their professional experience and judgment to decide how to most effectively use these assessment tools to elicit authentic assessment of student learning.

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## Appendix A

Table A

*Online Mathematics Assessment Resources*

| Resource | Bolster Academy, SOWISO | Google Forms | Lumen OHM MyOpenMath | Maple T.A., Möbius | WebWork MAA |
|---|---|---|---|---|---|
| **Compatibility** | | | | | |
| Standalone | x | x | x | x | x |
| Blackboard | | x | x | x | |
| Brightspace | | x | x | x | |
| Canvas | | x | x | x | |
| Moodle | | x | x | x | x |
| Sakai | | x | | x | |
| Other | LTI Integration | | LTI Integration | | |
| **Questions Types** | | | | | |
| Calculated | x | | x | | |
| Equation/ Expression | x | | x | x | x |
| Fill in the blank | x | | | x | x |
| Free response/ Essay | | x | x | x | x |
| Graph sketching | | | x | x | |
| Matching | | | x | x | |
| Multiple choice | | x | x | x | |
| Multi-part | | | x | | |
| Numerical | x | | x | x | |
| **Assessment Type** | | | | | |
| Math Specific | x | | x | x | x |
| Formative | x | x | x | x | x |
| Summative | x | x | | x | |
| **Feedback** | | | | | |
| Instant | x | | x | x | x |
| Computer generated | x | | x | x | x |
| Manually entered | | x | x | | |
| **Other** | | | | | |
| Generates statistical report | x | x | x | x | x |
| Question Bank | x | | x | x | x |
| Accessibility | Paid | Open-Source | Open-Source | Paid | Open-Source |

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NOTES FROM THE FIELD

# Mathematical Design Thinking in the Classroom through Graphic Art

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**ABSTRACT** This classroom study immersed high school geometry students in the creative and intellectually challenging design task of developing unique logos using mathematics and technology. The students applied and deepened their knowledge of transformations while using dynamic geometry software. One of the main aims of the task was to elevate student creativity and autonomy within the mathematics classroom while they engaged in mathematical design thinking to create their logos. The discussion provides insight into considering student work from a design perspective, which can offer students new ways to engage with mathematical concepts and make their thinking more explicit.

**KEYWORDS** *mathematical design thinking, logo design, modeling, prototyping, technology, geometry*

## Introduction

The current emphasis on modeling within the Common Core State Standards encourages students to engage in mathematical thinking by creating a model and learning from the mechanics of the model simultaneously (CCSSI, 2010). In doing this, students are actively discovering, learning, and applying relevant mathematics to the model they are creating. Additionally, this provides them with opportunities to develop problem-solving skills that are applicable outside of the mathematics classroom.

Bringing mathematical design thinking into the classroom through design tasks, similar to those in Project Lead the Way (2017), provides opportunities for students to engage with mathematics in a unique way. In these situations, students have the autonomy to learn and do mathematics in ways that make sense to them. Dynamic Geometry Software (DGS) provides an interface for students to engage with and mathematize real-world situations. Such technology can help students represent and model natural phenomena while making the mathematical concepts an explicit focal point. Technology also promotes mathematical habits of mind and normalizes productive struggle.

In this classroom study, I sought to understand better how students engage in mathematical design thinking and how teachers can best support students to engage in this type of thinking. I repurposed an artistic logo design task to incorporate geometric transformations to encourage geometry students to mathematize the task within DGS. In this way, they would naturally engage in the design process by applying the relevant mathematical concepts they have learned. As a result, all students actively engaged in mathematical design thinking to create their unique logos, each demonstrating varying levels of awareness of the design process.

## Review of Related Literature

### Mathematical Design Thinking

The Common Core Standards for Mathematical Practice (SMP) have brought increased attention to problem solving and mathematical modeling across the K-12 curriculum (CCSSI, 2010). The SMP explicitly mentions modeling, perseverance, and reasoning; these three actions fall within the construct of mathematical design thinking, and, more generally, what is known as *mathematical knowing in action* (Schön, 1992). Additionally, implementing strategies that intentionally bring design thinking into the mathematics classroom can positively impact how students approach rigorous problems (Chin et al., 2019).

Design thinking is defined as a type of knowing in action characterized by a constant interplay between the design and the designer's thinking about the design (Schön, 1992). In this model, each depends on the other, and it is not possible to separate the actions within the space from the ways of knowing. Adapting Dym et al.'s (2005) definition of engineering design to mathematical design, mathematical designers generate, evaluate, and articulate mathematical concepts or processes. In doing this, the designers attend to human objectives while abiding by the constraints of the situation to which the design will be applied. The term situation includes instances where a problem is being solved and where one has not been posed.

Incorporating mathematical design thinking into the classroom can help students become stronger problem solvers and bring lower-achieving students to a level equal to their average-achieving peers (Chin et al., 2019). Design tasks give students the autonomy to explore, create, and do mathematics at the highest level (Smith & Stein, 1998). Consequently, students can apply and develop mathematical concepts within a context that is driven by their interests in relation to the task's requirements.

### *Principles Guiding Mathematical Design Thinking in the Classroom*

While developing a task to engage students in mathematical design thinking, teachers must carefully consider their objectives and support students as they engage with this type of thinking. Kolko's (2015) principles of design thinking serve as a guiding framework for designing and implementing rigorous mathematical design thinking tasks within the classroom. These principles are:

1. Create models to examine complex problems.
2. Use prototypes to explore potential solutions.
3. Focus on users' experiences, especially their emotional ones.
4. Tolerate failure.
5. Exhibit thoughtful restraint.

(Kolko, 2015, p. 68 – 69)

**Modeling.** Modeling is a way to visualize situations and explore the facets and constraints of a problem, also known as the problem space (Kolko, 2015). In contrast, Kolko (2015) distinguishes prototyping as exploring and experimenting within a problem's solution space. While engaging in design thinking, students can use modeling to help them understand and represent the problem or task in ways that make sense to them, then use prototyping to develop solutions to the task. In K-12 mathematics, students regularly create models to represent the problem space so they can set up and perform a brief calculation to arrive at a teacher's anticipated answer. In this context, students are engaging in basic modeling but never reach the prototyping stage of design thinking.

**Prototyping**. Prototyping provides students a creative space where they have the autonomy to develop one or more unspecified solutions to a problem or task; students assume responsibility for determining the appropriateness and effectiveness of these solutions based on the criterion they develop. Prototyping is a nonlinear process marked by student engagement in cycles of design and validation (Fountain, 1990). By discussing prototyping as a cyclic process, we can classify students' actions within these cycles to allow both students and teachers to identify and emphasize the significance of these actions. Rothenberg (1990) explains that students can engage in prototyping with the intent to generate and explore ideas *(generative prototyping)* or to determine what aspects of their prototype are meeting their expectations and goals *(evaluative prototyping)*. Fountain (1990) classifies the prototypes that students develop but later discard as *throwaway prototypes,* whereas those that students develop and then modify as *evolutionary prototypes.* These terms are referenced later in the discussion of the task.

**Student's Experiences.** A safe classroom environment is essential to the success of mathematical design thinking tasks. Mathematics classrooms need to be places where students are comfortable taking risks and sharing novel ideas so they can embrace the freedom, challenges, and unknowns that occur during mathematical design thinking. *Rough draft talk,* defined by Jansen et al. (2017), provides students a space to share in-progress thoughts and ideas without the stress of evaluation. Teachers can engage students in rough draft talk by acknowledging and honoring their mathematical work and emotional experiences with mathematics at all stages of the design process. In addition, teachers can seek to understand a student's work from the student's perspective without imparting their perspective by practicing what is known as *mathematical empathy* (Araki, 2015). Rough draft talk and mathematical empathy normalize the nonlinearity of learning and elevate the design process over the product.

**Risk Taking.** Finally, students engaged in mathematical design thinking must manage the complexity of their preferred designs with the risks that are required to pursue and create these designs (Kolko, 2015). In the task described here, students who designed an entirely new logo took more risk than those who decided to recreate and modify an existing logo; those working with complex logos risked not having the time to finish their designs. Each student had to find a balance between complexity and the associated risk.

In determining how to apply mathematics in a new way to make their logo, the students can be seen as "doing mathematics" as defined by Smith and Stein (1998). They explain that these high-level tasks free students from the limitations of finding a solution and provide opportunities to engage with and discover mathematics while exploring the solution spaces to problems. At the same time, Smith and Stein (1998) also discuss that students may find themselves experiencing some level of anxiety "due to the unpredictable nature of the solution process" (p. 348). By acknowledging and normalizing the risks and anxieties that some students experience while engaging in design tasks, teachers can support students in managing risk and ambition throughout the task.

## Context and Task Details

### Classroom Context

A total of twenty-one high school geometry students engaged in this mathematical design task by constructing a logo in DGS using geometric transformations. I began with the expectation that students had some prerequisite knowledge of geometric transformations since they learned how to recognize, represent, and perform translations, reflections, rotations, and dilations earlier that year. We reviewed these concepts immediately prior to the task during a two-day introduction where students also learned to navigate GeoGebra, the DGS they would be using. It was important for students to explore the software and familiarize themselves with various tools, including how to perform each type of transformation needed for the design task.

### Design Task

I provided the instructions shown in Figure 1 for the logo design task. Students accessed the materials for the task (i.e., instructions, a reflection handout, and assessment rubrics) on a Google Classroom webpage. The instructions for the task are in Figure 1.

**Figure 1**

*Logo Design Task*

**Design your logo using transformations in GeoGebra.**

This is a time for your creative side to shine! Consider what your logo will represent. If you are not sure where to start, consider making a new logo for your favorite brand of shoes, clothes, fast food, etc. Bring it to life with color and design!

**Your logo needs to:**

1. Represent a company, activity, program, brand, or something else.
2. Use at least ***two transformations.***

The design task was crafted to engage students in mathematical design thinking, primarily prototyping. The entire task spanned three consecutive class periods, each of which were forty-nine minutes long. On the first day of the project, the students and I examined transformations in existing logos before they were given time to work on their own. Some students requested to use existing logos; this request was granted with the condition that they needed to modify the existing logo in some meaningful way. The students continued their work on the second day and submitted their work by the end of the day. Their submission included the logos and an accompanying handout where they discussed the transformations they used and how they engaged with the design process. On the third day, the students gave short presentations of their logos to their peers.

### Implementation

The logo design task challenged students to blend creativity with problem solving while also requiring them to apply mathematical knowledge of transformations to their design. I provided them with the autonomy to choose the transformations they applied to make their desired logo. Throughout the task, the students were challenged to combine their understanding of geometric transformations with a working knowledge of the GeoGebra software. The short timeline required students to monitor their progress while balancing their ambitions and the associated risk.

I spoke with students to uncover how they were making sense of the task, using technology strategically, and persevering through the task. My aim was to understand their thought processes as they engaged with the task in relation to the components of mathematical design thinking and the SMP (CCSSI, 2010). However, it can be difficult to capture a student's mathematical knowing in action. Students are not always fully aware of or able to articulate their thinking because it is a compilation of the many small and often unconscious decisions they make while engaging in design. Therefore, in an attempt to make their mathematical knowing in action more explicit, I asked students to record their design process and any discoveries they made on paper. This provided me with insight into their thinking and served as a guide for students when they presented their logo to their peers on the third day.

Throughout the three days, I ensured the focus was on students' ideas and work. To do this, I monitored student progress and generated and maintained discourse with students about their work, primarily by engaging the students in rough draft talk (Jansen et al., 2017). I was intentional in allowing students to remain the authority on their process and encouraged students to discuss their ideas with each other at various points. Students had a great deal of freedom to discuss their work with others throughout the activity.

#### *Data Collection*

All of the class sessions were videotaped to capture student work in progress, including the conversations between the students and between the students and myself. Students submitted a digital copy of their logo along with a handout. The handout that students completed consisted of the following four questions regarding their design process and their use of transformations:

1. What does your logo represent? How does it represent that?
2. For every transformation that you created, fill in the following table. You may use an additional paper if you need more space. (In the table, students named each transformation, recorded the pre-image, image, and explained relevant properties.)
3. How did you design your logo? Walk me through the process you used.
4. What 'aha' moments or discoveries did you have while creating the logo?

The presentations students made included some of the same information; however, the students were asked to explain specifically: 1) the theme of the logo, 2) transformations they used to create the logo, and 3) their favorite part of the task, something that was challenging for them, and anything they would change if they had additional time to revise their work.

## Narrative of Logo Design Project

In this section, I will discuss how students interacted with the design task. First, with a summary of students' engagement, followed by samples of student work and conversations that took place during the task.

### Progression of the Design Task

Upon receiving the task, students generally took three different approaches. One group began by exploring logos found through searching Google Images to brainstorm ideas for their design. This inspired some students to modify an existing logo; for others, it helped them consider attributes of logos in general. A second group took time to create a mental picture of what their logos would look like before using GeoGebra to create the logo. The third group immediately engaged in exploratory trial and error, playing with the various DGS tools to see what logos they could create. Video recordings from the class showed many students restarting their logos at least once on the first day. Some of these students reused similar concepts in their next design, while others pursued entirely different directions after discarding their initial work.

During the first day, many students indicated that they began creating their logos before knowing what their logos represented. The video recordings revealed that students had a general conception of what a logo is, and they used this conception to guide their design. Instead of creating detailed arrangements or including random shapes that did not connect to each other in some way, the students generally focused on creating a cohesive design of transformations that approached their general conception. As they developed and refined their logos over the three days, students seemed to use their personal experiences with logos to determine what their creations represented. Interestingly, these students then created their own companies that matched their logos instead of matching their logos to existing companies. Once they decided what their logos represented, they considered different aspects of their logos and how these aspects provided meaning.

During the task, every student focused on how to use DGS to create the logo to fit their self-imposed expectations of what the logo could be. Some students used the grid provided in the DGS to create transformations, while others utilized the functions in the software instead. The videos revealed that as the students worked, they were also curious about what their classmates were creating; they looked at each other's screens, shared ideas, assisted each other with the DGS tools, and listened to nearby conversations.

Towards the end of the first day and throughout the second day, students began adding finishing touches to their logos; these included hiding points, lines, and labels in the DGS to make the logo cleaner and adding color to various shapes to make the logo stand out. Although I showed the students how to include these details, they had the autonomy to decide whether to use them. The students were not required to add these additional cosmetic elements; some students decided not to include these while others were not able to due to the time constraints. From here, students used their completed logos to fill out the reflection handout and bounced ideas off each other as they figured out how to record their design process. I encouraged students to sketch the pre-images and images to aid them in writing about transformations.

On the third day, every student presented their own logo to the class. There was a great deal of enthusiasm and excitement regarding the visual appeal of the logos. While sharing, students typically noted an area of improvement in their presentation. The students were also given opportunities to ask presenters about their logos. I asked questions to prompt discussion of particularly interesting characteristics of the students' design processes as well as to highlight noteworthy ideas. Each student enjoyed the support, enthusiasm, praise, and applause of their peers at the conclusion of the presentation.

### Conversation About Student Designs

The sample work and conversations included below are chosen to highlight students' engagement with the design process. All student names in this paper are pseudonyms to protect student privacy. Two students made logos that incorporated their names; the data from their logos are included in the conclusions, but their logos are not pictured to maintain anonymity.

**Leith's Star.** The conversation in Figure 2 occurred on the first day of the project when Leith wanted to create a five-pointed star using rotations. In this conversation, Leith is trying to connect his conceptual knowledge of rotations to the procedural knowledge needed to transform a shape in DGS. Similar conversations about connecting the conceptual and procedural understandings of transformations occurred throughout, with a maintained focus on applying mathematics to achieve an aesthetic goal. Leith later assisted a peer in overcoming the same challenge.

**Alana's Egg Brand.** Alana had to make a decision about which transformation was the most appropriate to create rays for her "sunny side eggs" brand, which shows the "egg cracking at the break of dawn....which the sun is beaming (squiggles)" as she wrote in her reflection. The conversation and her work were captured on video

**Figure 2**

*Leith Determining the Angle of Rotation*

**Leith (L):** Asks how to rotate a shape to create a five-pointed star.

**Simon (S):** "There's a way to figure out the angle of rotation on a five-pointed star."

**L:** "How is that?"

**S:** "How many degrees are in a circle?"

**L:** "360"

**S:** "360 degrees, think of a rotation, you want to make it a complete circle, right?"

**L:** "Yeah"

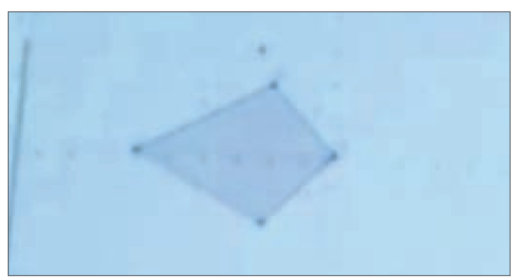
**S:** "So how many degrees would you make this rotation if it is 360 total?"

**L:** "360 divided by 5."

**S:** "Yeah"

**L:** (Thinking)

**L:** "So that would be, 70 something."



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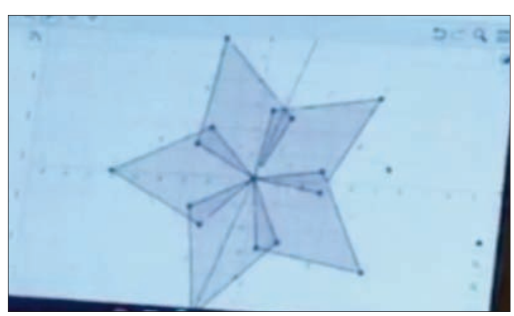

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**Figure 3**

*Alana Identifying the Appropriate Transformation*

**Alana (L):** "It's like an egg brand." (Top picture)

**Simon (S):** "To go on with the theme?"

**A:** "Yeah, and this is like the light, basically. But I need to make this (squiggles) better."

**S:** "Could you make one really, really good one and just use transformations to make the rest?"

**A:** "That sounds good."

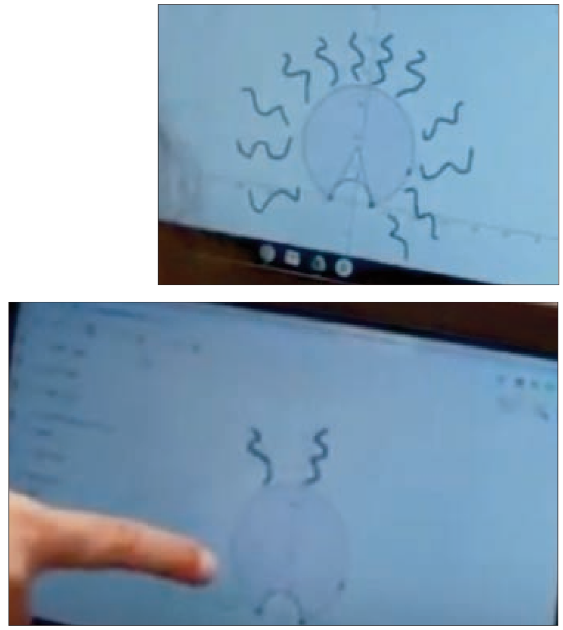
**S:** "Yeah? Which transformations?"

**A:** "Um, translation? … or rotation?"

**S:** "So if you translate, you start with a squiggle on the right, you could translate it down here, it would still be a squiggle here. So which one would be more sun like? Translations or rotations?" (Bottom picture)

**A:** "Rotations."

**S:** "Yeah!"



while she determined which transformation would be most fitting to create the sunbeams. This is another example of mathematizing a problem to reach an aesthetic goal.

In her presentation, Alana explained that "my favorite part was probably doing the squiggles. Um...I don't know why. Like all these are the same (pointing at the squiggles) except these two because I couldn't get them perfectly in the center, so, like, I had to redraw two of them. Or (actually) just one of them, and I reflected it." It is notable that the squiggles, although challenging, were her favorite part.

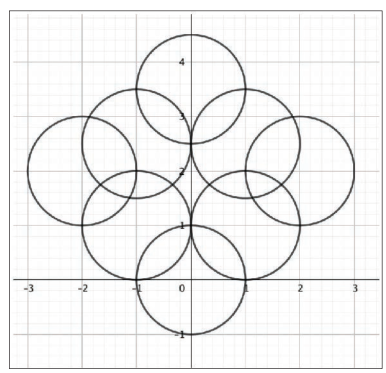
## Problematizing the Task

The logo design task emphasized aesthetics, with mathematics serving as the means to create them. In their reflections, most students recognized one or more mathematical problems or challenges they overcame to create the logo in DGS. Two student reflections appear alongside their completed logos in Figure 4. I chose these to illustrate how students used transformations explicitly and implicitly to create their logos.

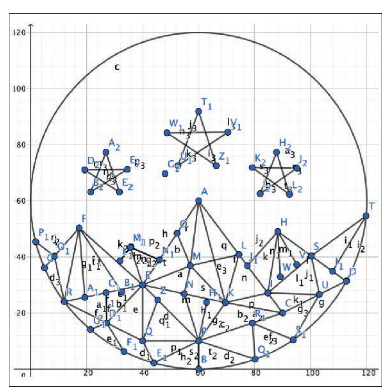
**Figure 4**

*Problematizing the Logo Design Task*

**Jacob:** "First, I had to find the best size for the circle. Then I had to find where to translate the circles to. This took the longest because it was hard to find where it would fit best. I figured out, and then once I finished the side, I reflected it to make the other side match."



**Haley:** "I made a circle big enough for the mountains I was going to put. I pressed segment lines, and from there I made the middle, so the main mountain, and from there I was making the 2 smaller mountains on both sides of the main mountain. After making those 3 mountains, I started making smaller triangles of different sizes facing different directions into the mountains so it wouldn't look so empty…"



## Results from Student Work

### The Role of Transformations in the Design Task

In their reflections, students identified and described the transformations they used to create their logo. From these reflections and video recordings of conversations, I identified why students used the particular transformations they did. Some students, including Leith and Alana, realized that they needed to use rotations to create the symmetry they were looking for in their logos. Others, including Jacob, found that translations and reflections helped them to move objects across the screen in different ways. Haley was one of many who applied dilations and translations to make larger and smaller figures. This demonstrates that students learned and understood which transformations were appropriate to reach their objectives in the aesthetics of their logo. Students did not use each transformation equally as shown in Table 1.

### Prototyping Within the Design Process

Students regularly engaged in prototyping throughout the task. The students' prototyping actions captured in the video were categorized in Table 2 based upon their intent, as either throwaway or evolutionary (Fountain, 1990), and their purpose, as generative or evaluative (Rothenberg, 1990).

One third of students created a throwaway prototype with the intent and understanding that they were simply generating ideas, and that their first design may not be their final logo. The students who began by researching logos or experimenting in GeoGebra engaged in generative prototyping; they were developing ideas of what to create before moving into DGS (Fountain, 1990). This is a stark contrast from traditional mathematical tasks where the objective is to obtain a correct solution. In these tasks, students are comfortable with experimenting and tolerating setbacks, which is consistent with design thinking (Rothenberg, 1990).

Due to the eventual evaluation of students' work (students were required to turn in a product that they presented and I assessed), all students naturally transitioned to evaluative prototyping by the end of this activity (Rothenberg, 1990). Many shared plans to revise their logos if they had additional time. Additionally, ninety percent shared at least one insight they had while creating their logos. Only two students denied having any insights; the videotaped conversations with these two students reveal otherwise; however, these students did not view those moments as significant in their reflections.

These results show that students engaged in prototyping as a method to help them incorporate transformations within their logos, even if their awareness of the design process varied. The data also revealed that students can engage in mathematics and graphic design simultaneously and that their logo designs influenced the mathematics they chose to incorporate within their logos.

## Conclusion

The Standards for Mathematical Practice establish the mathematical and critical thinking skills that all students should develop while learning grade-level content (CCSSI, 2010). The current modeling standard can be expanded to encompass all types of mathematical design thinking, including prototyping. By expanding this, we can create a space for the attributes and language of mathematical design thinking in the classroom and promote student autonomy and creativity in the learning process. Through mathematical design thinking, students can engage in cross-curricular activities from a mathematical perspective, and these activities can include but are not limited to graphic design.

Teachers who wish to integrate mathematical design thinking within their classroom must provide students

Table 1

*Percentage of Students Who Used Each Transformation Within Their Logo*

| Transformations | Reflections | Dilations | Translations | Rotations |
|---|---|---|---|---|
| Percentage of Students | 77% | 45% | 32% | 32% |

Table 2

*Percentage of Students Engaged in Each of the Prototyping Actions*

| Design Thinking Actions | Throwaway | Evolutionary | Generative | Evaluative |
|---|---|---|---|---|
| Percentage of Students | 33% | 100% | 52% | 100% |

with complex situations, contexts for students to apply and expand their understanding of mathematics. In mathematical design situations, teachers must create and maintain a classroom environment where students have ownership over their work and are comfortable engaging in challenge, intellectual risk, and productive struggle. When experiencing the design process through prototyping, students can learn that what initially appears to be a setback is instead a valuable conceptual gain that they may have otherwise not experienced.

Teachers can also do more to help each student be aware of and express their mathematical design thinking both verbally and in writing. One way of doing this is to teach the language of prototyping (generative, evaluative, throwaway, and evolutionary) so students can identify how their thinking and work fits within mathematical design (Rothenberg, 1990; Fountain, 1990). Student reflections provide insights, but their clarity and detail are dependent on students' abilities to verbalize or write their ideas after completing a design task and what the students view as significant. By recognizing the merits of their work and ideas through the language of prototyping, students may find themselves better able to record their design processes in detail.

Design tasks must also be developed and implemented to honor the creativity that emerges during the design process, which makes each student's work unique. Thus, incomplete and complete work must be able to exist side-by-side and elevated as equally valued contributions, communicating to students that their work is important and valued at all stages within the design process.

### Implications for Further Research

Research is needed to develop additional methods that engage students in mathematical design thinking and to further investigate the connection between mathematical design thinking and mathematical habits of mind. In conjunction, research of this type would provide additional support for teachers to expand on the mathematical modeling that already occurs within classrooms to include all types of mathematical design thinking, including prototyping. Consequently, this can strengthen student learning and understanding of mathematical concepts and deepen students' awareness of the design process. Such an expansion provides an opening for educators to develop diverse and innovative learning opportunities that empower students to be creative, take risks, and problem solve, thus supporting students to become active doers of mathematics.

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NOTES FROM THE FIELD

# Using the Sphero BOLT to Engage Students Mathematically

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**ABSTRACT** in this paper, we discuss the utilization of an innovative learning tool, the Sphero BOLT robot, in a 10-day algebra-based mathematics education course for graduate students. Students created routes for their BOLTs to travel and determined ways to measure the distance, rate, and time of their robots' movements. The student prompt, sample student work, class time considerations, and sample student-written reflections about the activity are detailed, in addition to implications and suggestions for teacher educators.

**KEYWORDS** *algebra, robots*

## Introduction

The use of technology in a mathematics classroom is a vital component of learning (NCTM, 2000), and the available instructional technology changes drastically from year to year. Utilizing STEM-based lessons with a mobile application-controlled robot, such as the Sphero BOLT (Sphero, Inc. 2019), to teach mathematics concepts is also becoming more popular (Dunbar & Rich, 2020). Robotic activities have led to science, technology, engineering, and mathematics (STEM) learning engagement improvements (Kim et al., 2015). The use of robotics has been shown to have positive effects on students' spatial ability (Coxon, 2012) and their interpretation of graphs (Mitnik et al., 2009). According to Ioannou and Bratitsis (2016), "problem solving, literacy, creativity, and motivation are positively influenced when children access technology in their learning environments" (p. 3).

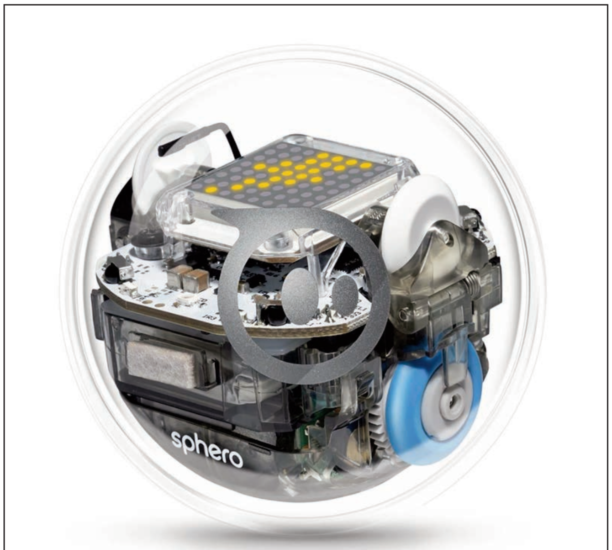
Robots are most commonly used with K-12 students during summer programs, after-school programs, elective courses, and robotic competitions (Altin & Pedaste, 2013; Barker et al., 2010; Larkins et al., 2013; Shepherd et al., 2019; Williams et al., 2007). In the classroom, the research on particular use of the Sphero BOLT is limited (Dunbar & Rich, 2020); however, the Sphero SPRK robot has been used in the kindergarten classroom for speed-related STEM activities (Ioannou & Bratitsis, 2016). Therefore, teacher educators should understand how such technology can be incorporated into their mathematics education courses so that they can prepare future and current teachers to use it in their classrooms. Additionally, there exists limited research on the use of robots in college mathematics classrooms. Therefore, the purpose of our work is to help fill this gap and show how the Sphero BOLT was utilized in a college mathematics education class.

Accordingly, we detail the use of pre-built BOLT robots in a graduate mathematics education class offered in the summer, which focused on teaching algebra topics covered in middle school to current and future mathematics educators. In this course, four graduate students, two pre-service and two in-service teachers completed two algebra-based projects using the Sphero BOLT. More specifically, in reference to mathematics standards, we detail the first classroom project (see lesson prompts in the Appendix) utilizing CCSS.MATH.CONTENT.6.EE.C.9: *Use variables to represent two quantities in a real-world problem that change in relationship to one another*. In particular, the graduate students were examining the relationships among distance, speed, and time. Students were required to keep one variable constant and look at the connections between the other two variables. Through this classroom episode, insight into how teacher educators and future teachers alike can utilize the BOLT in their respective classrooms are discussed.

### About the Sphero Bolt

Figure 1 shows an image of the Sphero BOLT, a grapefruit-sized sphere with a durable transparent shell that has been recently introduced to the public.

**Figure 1**
*The Sphero BOLT robot*




The BOLT relies on wireless charging, so there are no wires or openings on the exterior, and the motors and display are balanced, protecting the internal electronics. The pre-built, self-contained structure of this robot allows for teachers to utilize it in a variety of lessons; for example, it can be used in drawing and tracing activities in which the BOLT can be covered in wet paint and then driven or programmed to follow specific paths. Such an activity enables students to explore the movement of the robot in relation to their commands. The BOLT can be used at different grade levels since users can interact with the robot in a variety of different ways. One such way is the use of a touch-based mobile application that controls the robot's movement through blocks of code that can be dragged and dropped.

The drag-and-drop interface provides a programming experience similar to such programs as Scratch (2020). This allows students to drag "code blocks," which signify logical structures, input controls, and output controls into different arrangements, then they can insert values for pre-built variables to programmatically control actions. Using these pre-made blocks of code helps students focus first on the mathematical concepts at hand and second on the coding aspect of the lesson. Additionally, while the robot features 360-degree motion, the remote-control application also allows users to explore the concepts associated with labeling 0 degrees as directly forward and 180 degrees is directly backward. A student could then drive the robot using the application by dragging their finger from a designated center point in different directions of motion. In the lesson we describe, graduate students were able to explore how the robot could be used to teach mathematical concepts using both features of this interface.

### Class Specifics

The participants described in this classroom episode consisted of four graduate students enrolled in a 10-day summer mathematics education workshop at a university in the south-central United States. The students met for four hours each day for two weeks. The workshop for future and current teachers focused on middle school algebra-based mathematics standards. The use of technology in mathematics education was a central focus of the class. Therefore, the instructor demonstrated how to use graphing calculators, Vernier motion sensors (Bluetooth enabled sensors that measure distance/speed/time), algebra-based iPad-based applications, and Sphero BOLTs to teach algebra topics.

Before the work with robots, the students worked for approximately 60 minutes with Vernier's motion sensors, similar to Texas Instruments' calculator-based rangers (CBRs™), to develop a deeper understanding of movement in relation to distance/time graphs. The motion sensors were placed in front of students to measure student movement toward and away from the sensor. Graphs were produced on students' iPads and were discussed in detail.

To continue the discussion of distance/time/speed relationships, the instructor introduced students to the Sphero BOLT. The participants also had approximately 30 minutes of individual class time to familiarize themselves with the robot and its associated technology. The students could either use the application to move the robot with their finger and/or use blocks of code to move the robot along the floor. This exploration was open-ended; the instructor provided students with a great deal of freedom with respect to how to initiate movement of the BOLTs. This time was necessary for students to familiarize themselves with the technology as well as alleviate any anxiety associated with using a robot for the first time. The instructor circulated the room to answer any questions students might have about the application and robot. Students were able to use the Sphero Edu (Sphero Inc., 2019) application with ease and were able to move the robots once they were connected (Sample instructions on how to move the BOLT are provided under Step 2 in the sample lesson prompt at the end of the article.)

Of the two required lines of code, the first corresponded to rolling their robot in a certain direction with a designated speed and time. The second line of code made their robot stop. Even though there were only two lines of unique code required, students had to determine which mathematical values to input in the code each time to move the robot correctly. Students could use various devices, such as tape measures to measure distance, their phones to measure time, and protractors to measure angles.

## Lesson Details

Students then completed the BOLT-based project in groups of two over the span of 90 minutes. Group work and collaboration was encouraged throughout the activity. The instructor of the course has often utilized children's literature as a theme in her mathematics education classes to help future teachers see the value in using children's books to contextualize and teach mathematics (Jao & Hall, 2018). To demonstrate this and provide a context for the current lesson, the instructor utilized Kate Toms' (2009) book The Itsy-Bitsy Spider.

In the Sphero project, students listened as the instructor read parts of the story to provide context for the activity. Similar to the spider in the story who traveled out a waterspout and around town to get back home, the students were asked to create a story about a journey that the BOLT robot took. Although the students were free to produce paths of their choosing, they were required to incorporate at least three stops, as well as paths that were both straight and curved. These requirements were included to increase the cognitive demand of the activity.

Students created a path on sheets of poster paper and a corresponding storyline for their BOLT robot. They then utilized the Sphero block coding feature of the Sphero education application to generate blocks of code to move their robots along the path described in their storyline. The instructor pointed out to students that they only needed to repeat two specific lines of code, "roll __ degrees at __ speed for __ sec," and "STOP" to be successful in moving their BOLTS, but left open the opportunity for students to be more creative with these commands, such as including words and colors to their robot through different coding schemes.

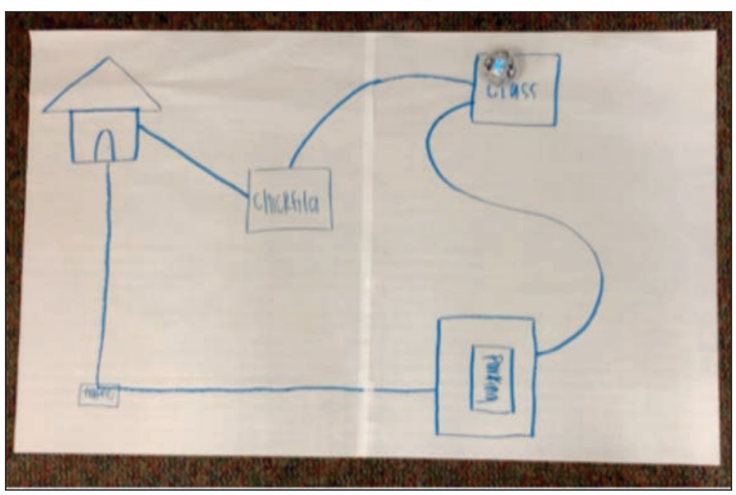
## Sample Work

Both groups created paths that detailed their robot going through a series of straight and curved routes. As an example, Figure 2 shows the path designed by one group, wherein their BOLT robot left home, got caught in traffic, parked, went to class, traveled to a restaurant to eat, and then returned home.

All students initially struggled with determining the correct values for direction, time, and speed that would make their robot move down each of their predetermined paths. Students also needed practice using tools such as protractors and tape measures. Students often moved their robot off-course and adjusted their values for direction, distance, time, and speed. After some exploration, both groups of students successfully produced working code. Perhaps more important to their learning and future teaching, the student groups were able to wrestle with relationships among the coding structure and the distance, speed, and time of the robot.

**Figure 2**

*Sphero BOLT on sample student route*



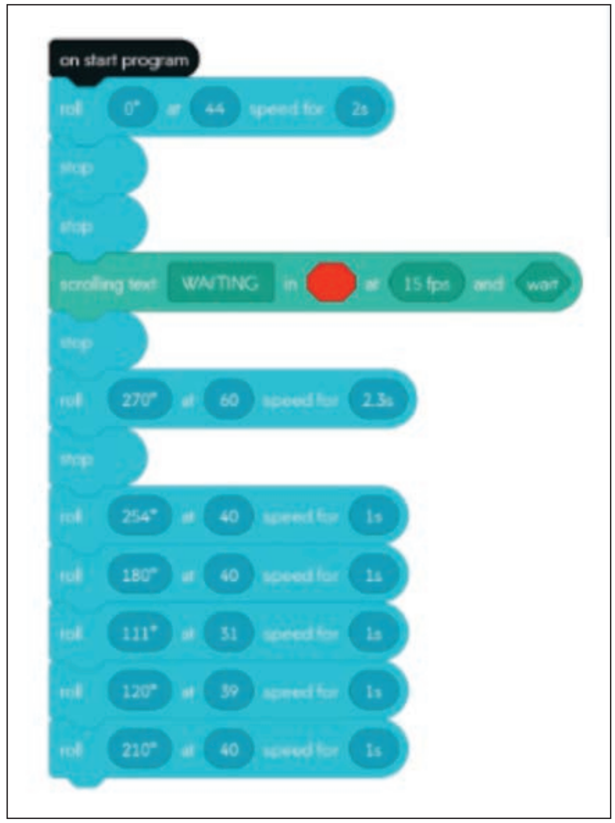

**Note:** This sample of student work was selected since they included some of the optional features, such as changing the color of their BOLT and writing phrases on their BOLT when at various stops.

Creating correct direction headings for the BOLT was difficult for some students. For example, a heading of 0 degrees (with the blue BOLT alignment light facing the person) would move the BOLT forward, and a heading of 180 degrees would move the BOLT backward. Students were more familiar with relying on traditional *x*- and *y*-coordinates for direction. However, the remote-control application for the BOLT robot requires direction headings to be in terms of angle measures and mimics how a person perceives their orientation in space.

For fun, the group that created the paths in Figure 2 also added color (e.g., red) to their robot and phrases (e.g., the word WAITING) that would scroll across the LED screen of the robot to simulate when their robot was waiting in traffic. Figure 3 shows a code created by the students.

**Figure 3**

*Sample student code*




## Student Reactions

In a written reflection about the activity, all students commented on how they enjoyed working with the BOLTs. More importantly, students also commented on how this activity could help their future students learn mathematical concepts such as distance/time relationships within a relatable common context. One student, who was uncomfortable with technology in general, commented:

> I am severely allergic to technology, so I was uncomfortable at first. I warmed up a bit during the first activity (playing around with BOLT).…Students could benefit from Sphero Activities…To see a 1D graph on paper vs. seeing a graph that they constructed through an activity that they created will solidify concepts and aid them in making connections to real-world activities. Really cool stuff. Thanks for stretching me today and getting me out of my comfort zone.

Another student commented on the usefulness that she sees in BOLT in the mathematics classroom:

> I loved today's activities!! The problem-solving skills required would be great to workout anyone's brain! I had fun working with the BOLTs while also using angles and speed variables to learn. I think this would be a good introduction to distance and velocity.

In a future lesson of this type, it would be meaningful to expand upon this students' comment and demonstrate how this activity could be used as an introduction to speed, time, distance, and velocity. Also, in the current lesson, the students engaged in making hypotheses and then testing them; however, in the future it would be useful to model for the current and future teachers how to help their students formalize such conjectures.

## Conclusion

Even though students initially struggled with creating correct distance, speed, and heading values to make their BOLTs move, they were ultimately successful. This productive struggle on the part of the pre-service teachers provides a context for them to use this meaningfully in their future classrooms. If given to an appropriate set of students with the correct support, the pre-service teachers could engender the same struggle and learning they experienced in this activity. Additionally, since only two unique lines of code are required to move the robot successfully, the aspect of this assignment that requires coding is rudimentary enough, and foreign enough to provide students who are new to coding an exposure to such concepts. Therefore, access to the mathematical and computer scientific concepts of such a lesson permit student engagement at a wide range of educational levels.

An important part of this lesson was the open-exploration time given at the beginning for students to familiarize themselves with the BOLT. This allowed them to explore intuitive notions of the robot's movement in comparison to the coding structure. However, in future lessons of this type, the instructors may include guided explorations, such as providing guidance for basic, horizontal, or vertical paths before having them move on to more complicated ones. The instructors should consider further the space in which this activity is taking place since it was easy for the BOLTs to collide with other objects or fall off tables. Another consideration for a future lesson may be the inclusion of a worksheet where students could document their trials and errors and provide explanations for their successes and failures. This could help students regulate their progress as well as have a valuable record of what mathematical learning was taking place. Lastly, prior to the lesson, a review of using appropriate tools such as protractors and tape measures may help all students focus on new mathematical concepts instead of recalling prerequisite ones.

An important aspect of the lesson that is particularly meaningful for future teachers is the modeling by the instructor of best practices for developing conceptual understanding. The instructors demonstrated how to generate and maintain discourse with students to help develop mathematical understanding, particularly with respect to the relationships of distance, rate, and time. The instructors modeled how to circulate the classroom during the activity and ask probing questions. Additionally, the instructors modeled how to ensure the activity was student-centered and student-driven by providing structure and guidance when needed, but also providing independence when needed as well.

This activity may interest mathematics teacher educators since it explores the use of an innovative technology in the teaching of mathematics. Although the students in the course were familiar with concepts related to distance, rate, and time, they were unfamiliar with how the technology introduced could be utilized to teach these concepts. As a result, students could authentically use these new experiences to provide engaging activities for their future mathematics students. Additionally, pre-service teachers benefit from engaging in innovative problem-solving activities where best practices are modeled by seasoned instructors. Activities such as the one discussed in this paper also provide evidence that an interdisciplinary approach is possible in certain settings. The current lesson provides future teachers with an example of how to combine innovative technology; mathematical and computer scientific conceptual learning; and even literature into one lesson.

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## Appendix

### Activity

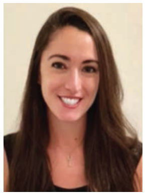
**Step 1:** Using the Itsy Bitsy Spider's adventure theme, create your own storyline and journey for a character of your choosing. Your character must travel to at least three places, with at least one path being curved and one being horizontal. Sketch your maze in the space provided. Once finalized, draw your path on the poster board and label each stop. (Remember, you must have at least 3 stops with at least one curve path and at least one horizontal path.)

**Step 2:** Now, using your Sphero BOLT and the Sphero Edu app, create code to move your BOLT through your maze.

*Directions for using the Sphero app:* Tap on the Sphero Edu icon. Go to My Programs, the plus icon, and choose Blocks code program type and the Sphero BOLT robot. Click Create. You can now drag and drop code onto your screen. (NOTE: The only two lines of code needed are to repeat two specific lines of code, "roll __ degrees at __ speed for __ sec," and "STOP" to be successful in moving BOLT, but you can get creative with your code by adding words and colors to your robot through different coding schemes.) When you are ready to test run BOLT, press START. The program will look for your BOLT to connect (make sure Bluetooth is on) and then run through your code. Have fun!!

In the space provided, list your lines of code and values. By each line of code, explain why you chose the specific degrees, speed, and seconds you selected.

# ABOUT THE AUTHORS



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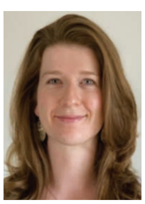

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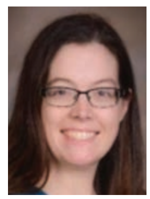

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