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A Century of Leadership in Mathematics and Its Teaching

Forward-Thinking Orientations for Mathematics Education

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AIMS AND SCOPE

The Journal of Mathematics Education at Teachers College (JMETC) is a recreation of an earlier publication by the Program in Mathematics Education at Teachers College, Columbia University. As a peer-reviewed, semiannual journal, it is intended to provide dissemination opportunities for writers of practice-based or research contributions to the general field of mathematics education. Although each of the past issues of *JMETC* focused on a theme, the journal accepts articles related to any current topic in mathematics education, from which pertinent themes for future issues may be developed.

JMETC readers are educators from pre-kindergarten through twelfth-grade teachers, principals, superintendents, professors of education, and other leaders in education. Articles appearing in the *JMETC* include research reports, commentaries on practice, historical analyses, and responses to issues and recommendations of professional interest.

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PREFACE

This issue is the Journal's third since the coronavirus pandemic hit in March 2020. Publishing a journal is no easy feat, especially during a global health crisis. We are grateful to our Editorial Team for the time and effort they put into improving the quality of the manuscripts. We commend our authors for finding time to write for this Journal during COVID-19 disruptions.

Amidst these challenges, the pandemic forced educators to reflect on and rethink longstanding practices in mathematics education. It is encouraging that many of the shifts in practice are centered around engaging students in developing their conceptual understandings of mathematics. Moreover, there is a movement towards more humane and equitable practices within mathematics education. With the birth of this Spring issue, in keeping with the significance of the season, we are hopeful that our community will use the pandemic as a catalyst for reform. Each article featured in this issue has a forward-thinking orientation for mathematics pedagogy.

In a dialogue with JMETC, Peter Liljedahl discusses his work on building thinking classrooms. During our interview, Dr. Liljedahl recounted a moment from early in his teaching career that converged with research interests in how teachers may engage students in thinking behaviors. This paper discusses important findings from decades of classroom-based research, shares ideas for facilitating a virtual thinking classroom, and highlights opportunities for creating equitable classrooms.

The following three articles are practice-based papers that focus on engaging learners in mathematical thinking. Authors draw from their classroom experiences to investigate how students at different stages might construct, extend, and apply their understanding of fundamental mathematics concepts. First, focusing on elementary mathematics, Dimmel, Pandiscio, and Bock address the limitations of discrete multiplication models in "Multiplication by Sunlight: How Can a Geometric Definition be Realized in a Physical Tool?" In response, they investigate the use of a physical continuous model—the *Sunrule*, an analog device—that uses sunlight to illustrate multiplicative relationships. This paper suggests an alternative for building students' early understandings of non-integer multiplication.

The next two practice-based articles explore the relationship between mathematical modeling and building positive mathematics identities. Imm uses storytelling as a catalyst for mathematical modeling in "Modeling as story-building and storytelling: Redesigning Algebra with Adolescent Girls of Color." Relaying episodes from a class for algebra "repeaters," she discusses how students used storytelling to engage with concepts such as rate of change. Next, Belin and Ferrell's students applied geometric concepts to understand a sociopolitical topic, gerrymandering. Students from their high school geometry classes used proportional reasoning, area, and perimeter to intuit measures of fairness for partitioning voting districts. The authors offer samples of student work and excerpts from student dialogue, then make recommendations for teachers who wish to implement the activities they outline.

PREFACE (Continued)

The last article in the main section, “Hyper-acceleration of Algebra I: Diminishing Opportunities to Learn in Secondary Mathematics,” reviews the literature on the acceleration and hyper-acceleration of Algebra I to Grades 8 and 7, respectively. Through their analysis, authors Galanti, Frank and Baker found a lack of evidence that hyper-acceleration prepares students for future success in mathematics. Moreover, they claim that hyper-acceleration perpetuates inequities in high school mathematics achievement. According to Galanti et al., the research on hyper-acceleration is limited. They advocate for further studies investigating the effects of taking Algebra I in Grade 7.

By the time we publish this issue, educators will be near, if not at, the end of their 2020-2021 school year. What a year this one has been! We dedicate this Notes from the Field section to educators, who have persevered through a tumultuous, often unpredictable school year to keep students learning during the COVID-19 crisis. This section features short entries from practitioners who reflect on their pandemic teaching experiences. Authors recount their strategies for engaging and supporting students amidst challenges of remote learning, hybrid learning, and personal hardships. Several look to technological platforms for facilitating communication and individualized support. During these disruptive times, others attend to the needs of their students by creating more humane classroom environments. Whether in adjusting the course objectives, implementing forgiving grading policies, or developing new classroom norms, authors in this section discuss the implications of their adaptations from this past year for teaching post-pandemic.

The articles in this issue do not offer a panacea for all mathematics education ills, but they provide examples of practice and thought relevant for enhancing mathematics education. We close this preface with a quotation from the National Council of Supervisors of Mathematics and the National Council of Teachers of Mathematics Joint Statement (2020): “We have the opportunity to be innovative and to think purposefully about addressing traditional/systemic structures, practices, and beliefs that have allowed inequities to persist” (p.15). As we take steps towards a stronger and improved mathematics community, we invite our readers to join us in keeping this call at the forefront of our minds.

Reference

National Council of Supervisors of Mathematics &National Council of Teachers of Mathematics (2020, June). *Moving Forward: Mathematics Learning in the Era of COVID-19*. https://www.nctm.org/uploadedFiles/Research_and_Advocacy/NCTM_NCSM_Moving_Forward.pdf

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Building Thinking Classrooms: A Conversation with Dr. Peter Liljedahl

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ABSTRACT This conversation is a slightly modified version of an interview with Dr. Peter Liljedahl, Professor of Mathematics Education. The interview was conducted by *JMETC* Guest Editors on March 8, 2021, and was based on Dr. Liljedahl's December 7, 2020 colloquium presentation "Building Thinking Classrooms: A Case of Results First Research" at Teachers College, Columbia University.

KEYWORDS *thinking classrooms, teaching online, equitable classroom practices*

Introduction

Dr. Peter Liljedahl is a Professor of Mathematics Education at Simon Fraser University in Canada. In this dialogue, he talks about his experience and research on classroom practices that provide opportunities for students to think deeply about mathematics. His new book *Building Thinking Classrooms in Mathematics, Grades K-12: 14 Teaching Practices for Enhancing Learning,* is a practical guide for teachers based on 15 years of research. Dr. Liljedahl discusses evidence-based practices from the book. He offers suggestions for how teachers can adapt his framework to teaching online during the pandemic and comments on building equitable thinking classrooms.

JMETC: Please get us started by sharing your favorite mathematics teaching memory.

Dr. Liljedahl: This is actually a memory that is critical in my growth and realization that what's happening in schools is not really that beneficial to students, sometimes. I was teaching a calculus course at a high school. I felt it was important to create and teach a calculus course for students who were most likely to be shocked by their experiences in calculus the subsequent year when they got to university. So, not the top-top students in the school—they were taking AP Calculus. This was the next tier of students. I had been teaching the students for several months, and I was moving through the curriculum and teaching in what I would say was a relatively traditional way.

We were about to start doing implicit differentiation of exponential functions, and they had all of the tools to be able to arrive at the process themselves. We had done implicit differentiation of trig. functions. They also had familiarity with exponential functions from the concurrent Math 12 course that I was teaching them. I thought I would do something different this morning. I wrote a question on the board, and I said, "Try to figure out what the derivative of this function is. Talk to the people sitting around you and see what kind of progress you could make." And they all sat there and stared at me. And you know, as a teacher, when you have that dead space when you've asked students to do something, and they're not doing anything, those seconds feel like hours. Finally, I said, "I have to go do some photocopying," and I left the room and stood outside to watch what was happening inside. Nothing was happening. They weren't talking. They weren't trying. They just sat there.

After about ten minutes of me standing out in the parking lot watching, I came back, and I said, "Okay, does anyone have any thoughts, is there any progress made on this task?" They just sat there and stared at me. After maybe a minute of this waiting, I said, "I have to do some more photocopying," and I left. And I stood out of the parking lot for another ten minutes and watched. And nothing was happening. So, I came back inside and asked, "Anything?" And they just stared at me, so I said, "Class dismissed." And their jaws just dropped. They were kind of in shock.

They came in two mornings later and the exact same question was written on the board at the front of the room. As they settled, I just said, "We're still on the same question. Talk to the people sitting around you, see if you can figure anything out, make any progress; I have to go do some photocopying." I left the room and stood out in the parking lot. I stood out there for 25 minutes before I actually saw them start talking and discussing. Then, after about ten minutes, I came back in, and I said, "Anything?" Now they had things to offer. For me, this was an awakening that we're doing students a disservice if we just chop curricula into small bite-sized pieces and feed it to them. Although I had given my students all the tools to be able to answer this question, they didn't know how to do anything on their own in a novel situation. They were waiting for me to spoon-feed them.

A few days after that, I announced that the upcoming unit test was going to be a take-home exam. They were to show up on Friday after school and would have the whole weekend to work on it. The last question was, "How many hours did you spend working on this exam?" When they came into class on Monday, they were in a really bad mood. But it was an interesting bad mood because they all sort of said the same thing: "That was way too hard. But we think we know what you mean."

My argument was, just because you don't know how to do something immediately, it doesn't mean you don't have the tools to do it. The actual purpose of this exam was to push them into that space. I asked questions that they had never seen before, all of which they absolutely had the tools to answer. They were allowed to use their notes. They were allowed to use a textbook. They had all the resources. But they were still going to have to apply these in ways that were going to require them to think. The average amount of time that they had spent on that exam was 20 hours. In the end, I gave them a choice of whether they wanted me to count that exam or not. The point was, I was trying to break them of this habit of just waiting for me to tell them how to do things.

I know what calculus looks like at university. It's three times as fast, and they cover more content, and there is no sympathy or empathy for a student who isn't keeping up. The goal of this experiment—my endeavor in this moment—was to get them to realize that moving forward in math is not about being shown 100 recipes on how to do 100 different things. It's about being shown a small set of fundamental principles and then figuring out how to apply them in novel situations. That was an eye-opening experience for me. It was, interestingly, not this experience that contributed to me starting this work on building thinking classrooms. But, as I was doing that work on building thinking classrooms, I did reflect back on that experience many times and realize that in some sort of a parallel-universe way, I was now working on the same problem I had encountered all those years previously.

JMETC: Where did your idea to wait and give them time to think over two days come from?

Dr. Liljedahl: I'd like to say that it was something deterministic, that I came into the lesson that day determined to do this. No; it emerged in the moment. I put up a question that I believe that they should be able to do, and I was met with silence. I don't know why I decided at that moment that I was not going to cave–that, rather than stand there and have that uncomfortable silence between us, I was going to leave the room. That was just to break that tension—that tension of them waiting for me and me waiting for them.

When I think back to my years as a student in university learning mathematics, I remember my third-year advanced algebra course (rings and fields): the instructor would give us homework assignments, and we had two weeks to submit them. Each homework assignment was usually six questions. It was not a large number of questions, and we had two weeks, but I remember that I would come home, and I would start to work on it, and after four hours of work, I would have no answers. But, I tried every question a little bit. I had made no progress towards an answer on any one of the questions, but when the two weeks were up, I always was able to submit a complete homework assignment. I realized in that time that what separated me from having an incomplete homework assignment to having a complete homework assignment was always just effort. It was effort and thinking, and persevering, and trying to find ways forward with the tools that I had.

I realized that this is what these students were missing. If they were going to university the next year and most of them were going to take Calculus I and II in their first year, this was what they needed. So, there were a lot of ideas happening at the same time. But I think, to be honest, at the moment, it was just organic.

JMETC: Would you please share some important highlights from your work?

Dr. Liljedahl: Building thinking classrooms is something that I've been working on for the last 18 years. It culminated in the publication of a book called *Building Thinking Classrooms in Math.* I think the clearest summary I can give is that this project is a reaction to a realization that in many classrooms I visit, students are not thinking. When I spent time at the beginning of the project in numerous classrooms, I gathered baseline data around how much time students spend thinking within one lesson. And, when I say thinking, I mean thinking in ways that we know students need to do in order to be successful in future math courses. What I'm not talking about is mimicking—that sort of regurgitation of what they've just been shown. I think there are many people out there who have had the experience of being mimickers themselves and then at some point hitting the wall in high school when the mimicking stopped working—when all of a sudden the questions became too intricate for a student to just be able to regurgitate something.

So, I was looking for thinking behavior as opposed to mimicking behavior. The baseline data showed that in a typical lesson—and this was true from kindergarten up, although primary always behaves a little bit differently—approximately 20% of students spend approximately 20% of the lesson thinking. The other 80% of the students spend zero time thinking. There's a small percentage of students who do a little bit of thinking when afforded the opportunity, but the rest of the students spend all of their time not thinking. They're mimicking, stalling, slacking, faking, but they're not thinking. Building thinking classrooms is a reaction to the non-thinking reality that I was seeing. This non-thinking behavior explains a lot of why we have troubles in mathematics around the world; it's not a uniquely Canadian issue or uniquely American issue. This may be the explanation: If students aren't thinking, they're not learning. Period! So, how could I get more students thinking, and for longer than they were in this baseline data?

That led me down a 15-year project of doing research into teaching practices. How can we do things differently that will afford and enable and require students to do more thinking within our lesson? Originally this work was chaotic, but was eventually organized around the core practices that every teacher does. There are 14 core practices that all teachers enact, and this emerged empirically from the spending time in classrooms. Every teacher uses tasks of some sort. Every teacher uses collaborative groups in some way. Every teacher has students do their work somewhere—the students have to do the work somewhere. We assign homework, we give notes, we do formative and summative assessments. We answer questions, we give hints and extensions, we consolidate lessons. And those became the variables around which we explored and organized this research. Are there practices within these 14 variables that will generate more thinking? It turns out there are.

In hindsight, it's not hard to understand why traditional teaching doesn't produce this kind of thinking. If we look back at the origins of public education, it was forged at the end of the first Industrial Revolution. It was built on some core principles, but it was really designed to create conformity and compliance. The practices that were established one hundred and fifty years ago have persisted in classrooms today. But our goals are different now; our goals are not to create conformity and compliance. We have these 21st-century learning goals; we're supposed to make students be critical thinkers and creative thinkers who are thinking outside of the box. We're supposed to address issues of equity and diversity. And yet, we're holding on to practices that were born in the crucible of conformity and compliance. How can we possibly expect those same practices to achieve these new goals? When we disregarded this convention and tradition and history and just looked for empirical evidence of how much thinking was being done, and how many students were thinking, what emerged were very different practices.

In a traditional normative classroom, students sit, the teacher stands. The teacher writes on the board, students write in a notebook. The teacher demonstrates how to do a task. The students replicate how to do that task. This is very conventional teaching and has been with us for 150 years. In a thinking classroom, students stand and write on the boards. They work in groups of three that are randomly selected. They work on tasks that they have not yet been shown how to do. Visually, this is radically different from how normative classrooms work. But there's more to it: There's lots of nuance in everything from how we give a task, to when we give a task, to how we arrange the furniture in our classroom, to how we answer questions, to how we assign homework, to how we consolidate the lesson. Every one of these things is very different. One of the things that emerged out of this research was that everything matters. How we do every little thing in a classroom makes a difference to whether we promote thinking or enable non-thinking behavior.

JMETC: When we publish this issue, teachers will still be teaching during the pandemic. What advice do you have for building a thinking virtual classroom, particularly when it comes to being intentionally less helpful—one of the optimal practices that you recommend for building thinking classrooms?

Dr. Liljedahl: We've had a year to play with this. I can't believe it's almost exactly a year now. Just as my book was ready to be released, which is 15 years of research in face-to-face classrooms on how to teach in face-to-face classrooms, we go online. So, we spend time thinking about and exploring and experimenting with which of these practices are still transferable from face-to-face to the online environment and which ones are either not at all applicable or need to be adapted. Surprisingly, a lot of things transfer with minor tweaks or with no tweaks at all. And I think the reason for this is that, fundamentally, we're still trying to achieve the same goal. If thinking is a necessary precursor to learning, whether we're online or face-to-face, we have to find ways to get students to think. If we want students to think, number one, we have to give them something to think about. We have to give them a thinking task. What defines a thinking task? Thinking is what we do when we don't know what to do. So, we still have to give a task where they don't know what to do. Which means we can't pre-teach how to do it. We give them something to think about, we give them someone to think with—so we're still going to put them in groups to do that thinking. We need to give them somewhere to do the thinking. So, we need to give them a space where they can share notation and share representation.

This is true in a face-to-face classroom and it's also true in an online setting. So, how are they different? In a face-to-face classroom, we have such ubiquitous access to resources that I think we've always taken for granted. Whether students can sit on the floor, or they can go and get manipulatives, or they can write on a small whiteboard or they can use technology, there's that ubiquitous access to resources when doing a task. All of a sudden, when we went online, we lost a lot of resources, and that ability to use resources to do a task. So, one of the things that we have to do is to make sure that the task we assign matches in some way the collaborative space that students can work in. If our online environment allows students to talk to each other through Zoom or through some other visual collaborative space, like you and I have right now, then that sets up a kind of situation that allows for certain tasks. If, in addition to this, they have a representational space where they can draw and show their notation, that opens up more possibilities for tasks. If the only thing you and I have is some sort of online asynchronous forum, that determines a different type of task that we can use. So, the digital space in which the teaching and learning is happening, in many ways, has to be taken into consideration when we're picking the thinking tasks. I think a task I would use in a face-to-face classroom may not work in an online classroom, so I have to be careful about how I pick my thinking task.

In a face-to-face classroom, we learned that the optimal group size is three—two if it's in a primary lesson. This has to do with the idea that in order for a group to be generative, there has to be a balance of both diversity and redundancy—this comes from complexity theory. Redundancy is the common knowledge or the prerequisite knowledge that we share. That knowledge is necessary for you and I to even begin to communicate. If you're talking about race cars in French and I like talking about fishing in German, we're not going to have a lot to talk about. We need to have something in common in order to just kick start the conversation. But if all we have is what's in common, then there's no advantage to working in a group. We need to also have diversity, which means that I have to know and think about things that you don't, and vice versa. It is this diversity that allows a group to be greater than the individual. Why are groups of three so optimal? Because the bigger the group gets, the harder it is to have redundancy. But the bigger the group gets, the more diversity you get. The smaller the group, the more likely you are to have redundancy, but you're going to have less diversity. So, groups of three seem to have this perfect balance between redundancy and diversity. The problem is that when we put three students online in a collaborative space, one of them has technology issues, the other one does not want to turn on the microphone or their camera. All of a sudden, the diversity just gets depleted. We found that when we went online, we needed to increase the size of the group to five. We needed to artificially keep that diversity up because online spaces are diversity-depleting spaces. When the group starts to really sync, and students become more open to having their microphones and their cameras on, and they solve all their technology issues, we can start to go back down to groups of three. But, initially, we need larger groups.

Then, we need a collaborative space; we need a space where we can actually share notation. Talking and waving our hands around a lot is good for some tasks, but once we get into more challenging mathematics, we need to actually be able to draw graphs, solve equations, and share notation. So, how do we find those spaces?

Well, things like Zoom have a collaborative whiteboard that we can use. There's a number of third-party digital whiteboards that can be used for groups to collaborate, and my favorite is Jamboard. The nice thing about Jamboard is you can use it with any collaborative space, whether we're in Zoom or Google or Teams. Even if we're on a phone or Facetime or Skype—haven't heard that term for a long time—we can open up a shared Jamboard and work on that. So, those are some of the big things that needed to be changed.

Now, you asked, in particular about being deliberately less helpful. In a face-to-face setting, being deliberately less helpful is actually a move that necessitates that the group start to collaborate more with each other—it's a way to increase knowledge mobility. In a traditional setting, knowledge transmission always happens from teacher to student. If we have 30 students sitting there, all knowledge transmission is happening in one-way lines from the teacher to each student. In a thinking classroom, we have these groups of three working together, so knowledge is transferring between those three students. But they're standing next to another group of three and we need knowledge to move between them as well. We need knowledge to move between this group and the group that's on the other side of the room. This is possible because they're writing on a vertical whiteboard. You can see what they're doing. We need knowledge mobility to happen. But knowledge mobility doesn't happen if the teacher holds on to this idea of always being the source of knowledge. If every time a group puts up their hand, and you go in, and you answer their question or give them the next question, they don't need to tap into the knowledge that's in the room. So, this idea of being deliberately less helpful is a way to force them to capitalize on the knowledge that exists around them. And there's other nuances to that practice.

What's the equivalent of that when we go online? Well, it turns out if we're using Jamboard, for example, the students can actually flip between the frames on Jamboard to see what other groups are doing. By being deliberately less helpful, I can force them to do that, but there is another element to this. In a face-to-face, thinking classroom, a group of students who gets stuck on a task will very quickly just look over their shoulder to see what a group is doing next to them, and maybe they'll see "Oh, they've made a graph" or "I like the way they organize their table" or "Huh, that's interesting, what they're doing there," and if it's interesting enough, and they don't understand it clearly, they may turn and talk to the other group.

What's the online equivalent of being able to just look over your shoulder? It's hard to do. I can't jump from one Zoom room to another as a student. I can go look at the Jamboard, but still, I can't communicate with them; I can't hear what they're saying. So, to help with this, we created another type of document. It's called a knowledge feed, and it's usually just a Google Doc that all groups can access. On this Google Doc, both the teacher and students can paste screenshots , so a group might say, "We found an interesting representation" or "We think we have a nice graph," and drop it in. Or the teacher might take a screenshot to capture what a group is doing and say, "Look at the way they organize their data." This knowledge feed is very much like a Twitter feed—it's this constant stream of ideas and knowledge. This needs to be there in order for the teacher to even begin to be deliberately less helpful. So, when a group asks a question, the teacher can just say, "Did you check the knowledge feed?" Rather than wait for the teacher all the time, if a group is stuck or ready to move on, they can go to the knowledge feed to get an idea, to get some inspiration, or get the next question.

JMETC: Could you talk about how your ideas for building thinking classrooms also help to build equitable classrooms? Do these ideas work for all classrooms and for all students?

Dr. Liljedahl: Recently, people have been talking about thinking classrooms through the lens of equity. I've received a lot of accolades for my attention to equity, and of course, I believe equity is important. But, equity was not one of my research questions when I was pursuing and building thinking classrooms. My research question was, can I get more students to think, and can I get them to think for longer? I think equity has become a byproduct of this process. This doesn't sound very nice because I should have been paying attention to that deliberately, but that wasn't my goal at the time.

So, the question is, where are the opportunities for equity to be created within the thinking classroom? And there are a number of them. I think that the most critical example is how students are grouped. Students are grouped randomly. And not just randomly, but visibly randomly, so that the students can see that there's randomness at play—for example, they might come up and pull a card, and all the fours are going to be together. When we interview students about this, they say that if the teacher is willing to throw this to randomness, then it means we are all the same. What we're communicating to students when we assign groups randomly is that nobody in this room is so special that they have to be paired with some other person, either in order to support them or to be supported by them—we are confident enough in all your abilities that it doesn't matter who you work with. We're all the same. I think this is one of the things that people are attending to when they start to say things like "Thinking classrooms helps with equity." I'm not claiming that because I haven't done the research on it. But, I think this is what they're attending to when they're making these sorts of comments.

There are other things, and again, I'm only talking to you through the lens of what I hear from students. One of the things around the practice of how we answer questions is that students ask three types of questions. They ask *proximity* questions which are the questions they ask just because the teacher happens to be close. They ask *stop-thinking* questions—"Is this right?" and "Are we doing this right?" And they ask *keep-thinking* questions which are extension questions or clarification questions to enable them to get back to work. We spent a lot of time trying to pay attention to the types of questions students asked and then gave an appropriate response based on the type of question. What we found was that the absolute best response when a student asked a proximity question or stop thinking question was to smile, nod, go "Interesting," and walk away.

When we interview students in the first few days of experiencing this kind of response, they are generally irritated—"The teacher just walked away, they're not answering our question." But when we interview students three weeks into this experience an interesting phenomenon occurs. We ask the students, "What just happened? You asked a question and the teacher just walked away what does that mean?" The students say, "Well, that means she thinks we can do it." And then we ask, "Well, do you think you can do it?" The students say, "Yeah, we probably could; she's been right before." So, it's again this idea that without any bias, the students are picking up on this idea that the teacher believes that they can do it.

The ninth practice is how we give hints and extensions. For this, the research was heavily informed by Mihaly Csikszentmihalyi's theory of flow and the idea that we have to keep students in a balance between their ability and the challenge of the task. We keep them in balance in how we give hints and extensions. As a group's ability increases, we extend the task, we increase the challenge of the task. And if the challenge is too great for their ability, we give them hints in order to maintain that balance. This is actually a form of differentiated instruction because the teacher is catering to the explicit and immediate needs of a group, whether or not that need is to increase the challenge or increase the ability. They're working with the group to see where they are in that moment, then deciding whether or not that group needs an extended task or if it needs some help.

But this is a very different form of differentiated instruction from what we typically see as differentiation. In this form, all groups start on the same task, and what's differentiated is the pace through which they move through a sequence of tasks and the degree to which they are supported in that venture. The differentiation happens in real time, based on real data of how that group is functioning in the moment. Classically, differentiation is based on an assumption of how an individual or how a group will perform that day and, based on that assumption, the student or the group will be given a different task. So, both forms of differentiation are about meeting students where they're at, but classic differentiation is predicated on assumption and bias and presupposition. What do we think this group will do here? What do we think this individual is capable of? And one of the things that we kept seeing in thinking classroom research was, what a teacher *thinks* a group is capable of is rarely reflective of what they're actually capable of. If we put three students together, often the teacher will say, "That group's going to struggle," and then ten minutes into it, this group is leading the class—they're tearing through the material. So, I think again, this is an equity move in the sense that, rather than assume that we know how a student is going to perform, let's actually see how they perform—and then differentiate our support around that.

But, again, I give the caveat that I'm no specialist in equity. I'm just echoing what I think other people who are experts on equity are seeing in the thinking classroom.

JMETC: Even though it's not something that you set out to do initially, it's important that you've given it some thought. Thank you for sharing.

Dr. Liljedahl: There's a new podcast that was just launched called *The Sum Of It All* (https://anchor.fm/sdcoe-math). It's two people in San Diego, and they're going through the book chapter by chapter, but one of the questions they ask for every chapter is, what are the implications around equity for this chapter? I really like listening to what they have to say because they are asking these hard questions.

JMETC: What plans do you have for building on this work? What's next?

Dr. Liljedahl: I'm working with the publisher to produce a book on building thinking classrooms in atypical settings in math. Online is one of those atypical settings. I think parts of it are going to be with us for a very long time. Also, the idea of hybrid classrooms, blended classrooms, working in a one-to-one situation, or a teacher working with a single group to support unfinished learning. I'm working on what thinking classrooms can look like in these settings.

For many years now, I've also been exploring what thinking classrooms look like in subjects other than mathematics. I've done a lot of work with English teachers, social studies teachers, history teachers, science teachers. How does the framework change when we move into other subject areas?

There's also a real thirst for thinking tasks to go with thinking classrooms and looking at those both from a non-curricular perspective and a curricular perspective. So, there's always something on the go.

JMETC: I'm going to round up by asking you, is there a question that I should have asked you but did not?

Dr. Liljedahl: I think an important question is, how do teachers receive this research or this book? Because the research can come at them in lots of different ways—it could maybe be part of a workshop, they could read the book, they could have bumped into an article on it. Or maybe they encounter a teacher who has encountered the book, and so now they're interacting with a teacher that is excited about what they're playing with and so on. I think there are elements of the thinking classroom research that teachers immediately agree with. I also think there are elements of the thinking classroom that really challenge us as educators because the results in some ways are very contradictory to what we were trained to do, what we were prepared to do, what we ourselves experienced as learners for so many years. It may challenge our very foundational beliefs about what is mathematics and what is important in mathematics. But I think that this is a healthy part of exploring our own practices—to feel challenged.

The thinking classroom framework—these 14 practices and the sequence that it comes with—is just that—a framework. It emerged out of research. I think this framework is enhanced when teachers add their own personality to it. So, it's not a prescriptive sort of recipe—do this, then do that—it's a collection of ideas and a collection of results, and teachers, of course, have to bring their own identity to those results. I often say at the end of a workshop that everything I've told you is guaranteed to fail, and guaranteed to fail in ways that are painful for everybody involved, unless you want it to work. If a teacher wants it to work, then they will find ways to make it work.

I often get asked what is the absolute best quality that a teacher can bring to thinking classrooms. I always have the same answer—if a teacher is willing to be seen as fallible in front of their students, I think that they have all the qualities they need to become a great teacher. If a teacher is willing to be seen as someone who's fallible, who's willing to try things and have them not work, then they have the courage to pursue their own self-study in improvement. The book is just going to help them on that journey.

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Multiplication by Sunlight: How Can a Geometric Definition be Realized in a Physical Tool?

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ABSTRACT Physical models for exploring multiplication are fixtures in elementary classrooms. The most widely used physical models of multiplication are collections of discrete things, such as Cuisenaire rods, Unifix Cubes, or Base-10 Blocks. But discrete physical models are limited in the products they can represent. By contrast, pictorial models, such as number lines or area models, are continuous and thus represent a broader range of products. However, pictorial models are limited in how they can be manipulated. The discrete/continuous divide across physical/pictorial representations of multiplication frames the overarching design problem that motivated our work: How could a physical, manipulable tool realize a continuous model of multiplication? This is a significant problem because, to our knowledge, there are no examples of physical models of multiplication that offer the plasticity of pictorial models. We describe one such model here—an analog technology that we refer to as a *Sunrule*. We explain the design of the device and report an initial instructional activity where pre-service teachers explored the device in groups.

KEYWORDS *multiplication, sunlight, physical models, sun shadows*

Introduction

The gradual accumulation of knowledge about multiplication in school follows a known trajectory. Basic facts, such as times tables, are memorized. From there, students learn a collection of algorithms for calculating the products of different types of numbers, including multidigit integers, decimals, and fractions. One takeaway for students is that multiplication is a collection of rules that apply in different circumstances. Indeed, elementary students and teachers alike tend to have procedural dispositions toward multiplication (Fuson, 2003; Hiebert, 2013; Lampert, 1986).

To help children develop a deeper familiarity with multiplication, some teachers rely on physical or pictorial models (Kosko, 2019). By *physical models* of multiplication, we mean tangible, graspable manipulatives such as Unifix cubes, Cuisenaire rods, and Base-10 Blocks. Such models are fixtures in elementary classrooms because it is believed that "children need opportunities to work with objects in the physical world before they will be ready to work with pictures and other representations" (Reys et al., 2014, p. 25). Physical models are generally *discrete* because they are collections of fixed quantities (e.g., a set number of cubes or blocks, a set of rods of specific heights; Kosko, 2019). By contrast, *pictorial models*, such as area or number line representations, are generally *continuous* because they are not limited to a fixed set of pre-determined things. Elementary teachers generally use discrete models to represent multiplication (Kosko, 2019).

While physical models offer tangible representations of multiplication, discrete objects are limited in the types of products they can represent. Meanwhile, continuous pictorial models can represent a set of unlimited products; however, such models cannot be investigated tangibly to the same extent as their physical counterparts. The discrete/continuous divide across physical/pictorial representations of multiplication frames the overarching design problem that motivated our work: How could a physical, manipulable tool realize a continuous model of multiplication? This is a significant problem because, to our knowledge, there are no examples of physical models of multiplication that offer the plasticity of pictorial models. We sought to explore how a model of multiplication that combined the tangibility of a physical thing with the flexibility of a picture might create new opportunities for exploring multiplication.

Design of the Sunrule

To model continuous multiplication using a physical object requires some method for increasing and decreasing lengths. We refer to this as the *variable length* design problem. The historical solution to this problem was the *slide rule*, an arithmetic aid that reigned from the 17th century until it was abandoned for electronic calculators in the 1970s (Tympas, 2017). Slide rules are ingenious, powerful devices that deserve a place in mathematics classrooms. However, as their logic of use is circumscribed by the theory of logarithms[1], they are not suitable for helping elementary-age students explore multiplication[2]. How else could adjustable lengths be used to model multiplication with a physical tool? Our answer to this question was based on McLoughlin and Droujkova's (2013) diagrammatic definition that frames multiplication as *continuous directed scaling*—i.e., the length of one segment is a positive or negative multiplier that scales the length of another segment in the positive or negative direction. Their diagrammatic definition of multiplication was inspired by Hilbert's treatment of segment multiplication (Hilbert, 1999). Dimmel and Pandiscio (2020) illustrate how the product of two segments can be constructed with a compass and straightedge.

To design the *Sunrule*, we used sunlight as a straightedge. According to Decamp and Hosson (2012), sunlight offers a readily available, renewable, and abundant supply of naturally occurring parallel rays. We recognize that the sun's rays are not entirely parallel. Still, over small distances, sunlight is several orders of magnitude more parallel than other real-world examples of parallel lines, such as railroad tracks. The sun's parallel rays mean that the height of any shadow-casting object is proportional to the length of its shadow, and, for a given altitude of the sun, this proportion is the same for all object-shadow pairs (Douek, 1999). Thus, multiplying numbers, in general, requires control over the position of the sun. We refer to this as the *variable altitude* design problem.

While the sun cannot be moved, there is a solution to the *variable altitude* problem: We can change the apparent altitude of the sun by varying the angle of inclination of a surface onto which shadows are cast. By increasing the angle of inclination of a surface, we decrease the lengths of any shadows falling upon it; by decreasing the angle of inclination, we would increase the lengths of those shadows. Thus, by varying angles of inclination, it is possible to control the apparent altitude of the sun from 90 degrees (i.e., directly overhead, no shadow) to 0 degrees (i.e., sun on the horizon, undefined shadow). Below, we explain how the *Sunrule* can be used for multiplication and illustrate how it solves the *variable length* and *variable altitude* design problems.

Multiplication by Sunlight

The *Sunrule* models multiplication by casting shadows. It is not a combination of a sundial and slide rule; however, the name is apt because it combines essential elements of each tool (e.g., gnomons[3], adjustable scales) in novel ways. A *Sunrule* consists of two ruled gnomons affixed at right angles to a ruled surface—i.e., the *shadow plane.* The device is independent of any particular choice of unit (e.g., inches, cm, mm), in the sense that the unit could be of any height, and that unit determines the other rulings. It is sufficient for the unit gnomon and the adjustable gnomon to be ruled in equal increments; the surface could have other (equally spaced) increments as rulings. We have found it convenient to use the same unit increment for the gnomons and the shadow plane, though we recognize that this is a design parameter that could be varied and explored.

For the *Sunrule* shown in Figure 1, there is a longer gnomon on the bottom and a shorter gnomon on the top. The shorter gnomon functions as a unit length. The unit length and the factor by which its shadow was scaled define a multiplier; in this case, that multiplier is 3. The

1 A slide rule's sliding scales are ruled in logarithmic increments, which allow products (multiplication) and quotients (division) to be expressed as sums (addition) and differences (subtraction).

2 This is a comment on the elementary mathematics curriculum, not a statement about the capacity for children to learn the theory of logarithms.

3 This is the name for the part of a sundial that casts a shadow.

Figure 1

A Sunrule *constructed by elementary teacher candidates*

PHOTO BY MEG PANDISCIO (2020)

device is positioned so that the length of the unit shadow extends to 3 units. The height of the longer gnomon can be adjusted by sliding it up or down; the height of this gnomon specifies the multiplicand, which is 4, in this example. The product, 12, is given by the length of the shadow of the adjustable gnomon. A video demonstration of how to construct a *Sunrule* is available at: https://tinyurl.com/wnjky5e4

The *variable length* and *variable altitude* problems solved by the *Sunrule's* design prescribe two movements that can be used to transform multiplication problems. One movement is that the angle of inclination to the sun can be varied by tilting the device up or down. Note that the gnomons in Figure 2 are in the same positions as those shown in Figure 1. That is, we left the multiplicand gnomon at four units. But we tilted the device so that the length of the shadow of the unit gnomon was two units. With that, the angle of inclination of the device toward the sun increases. This increase in inclination decreased the length of the unit gnomon's shadow from 3 units to 2 units; therefore, the multiplier changes from 3 to 2. As a result, Figure 2 shows 2 × 4 = 8.

The *Sunrule* provides a material context that models multiplication as a scaling operation. Because the tilt of the device can be varied with continuous movements, the device multiplies fractions as readily as integers. This is a potentially significant affordance because fractions

Figure 2

A Sunrule *that shows 2 × 4 = 8*

PHOTO BY MEG PANDISCIO (2020)

Figure 3

A Sunrule *that shows 1.5 × 3 = 4.5*

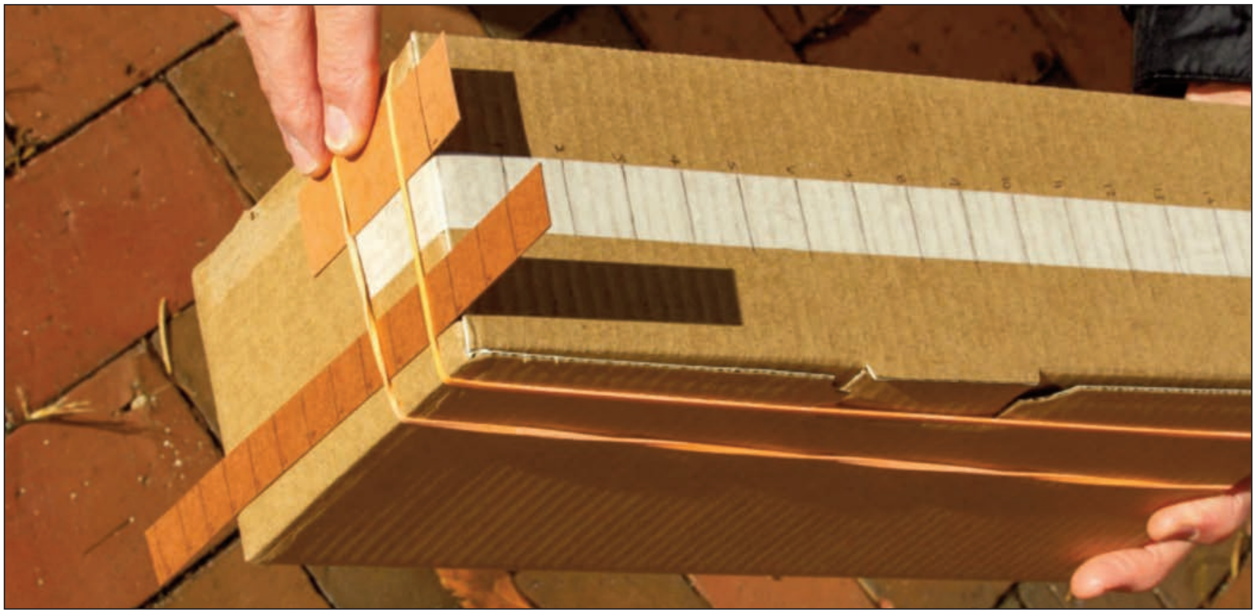


PHOTO BY MEG PANDISCIO (2020)

Note. In this image, a *Sunrule* has been inclined so that the length of the unit shadow is 1.5 units, the height of the multiplicand gnomon is three units, and the adjustable gnomon's shadow is 4.5 units; hence, 1.5 × 3 = 4.5.

are an endless source of difficulty for students (Sidney et al., 2019). Figure 3 shows 1.5 × 3 = 4.5. This is just one example of a fractional product defined between 3 and 4; other products could be modeled by slightly changing the angle of inclination. The multiplier would be slightly less than or slightly more than 1.5, and the length of the multiplicand shadow would vary proportionally. It is not any particular fractional product but rather the ability to move between fractional products—in an arithmetic analogy of the continuous transformations that can be used to explore dynamic geometry diagrams—that is potentially significant.

We do not propose that the *Sunrule* should replace other models for teaching multiplication; however, because it is a physical model that allows for products to be explored through continuous variation—by (a) varying the height of the adjustable gnomon and (b) varying the angle of inclination of the shadow plane—it warrants investigation. In an initial effort to gauge its pedagogical value, we facilitated an activity with elementary teacher candidates.

Initial Teaching Activity with Elementary Mathematics Teacher Candidates

During Fall 2020, the second author taught two sections of an elementary mathematics methods course, each with five students enrolled, that met on different days. The *Sunrule* investigation was planned as a two-lesson activity. For the first part of the activity, students worked with the second author to build *Sunrules*. Students were told that the device was for a mathematics exploration and that it needed to be used outside on a sunny day. Students were not told that the *Sunrule* was designed to model multiplication because we were interested in how students would explore and make sense of the device. Both sections of the course completed the first part of the activity; however, one group did not complete the second part because of inclement weather.

For the second part of the activity, students explored their *Sunrules* outside in two small groups. For safety, they wore masks and followed social distancing protocols. We used fixed video cameras to record the activity of each group. The second author moved back and forth between the groups to facilitate their explorations of the device. Using a semi-structured protocol, he provided directed guidance to the groups of students. An example of a directed question was: What are the ways that the lengths of the shadows of the gnomons could be varied? The purpose of this question was to help students identify the two movements through which the shadows could be varied to model products. The second author posed questions from the protocol to each group, as needed, to keep the students from getting stuck and guided them toward investigations of its mathematical affordances (i.e., a physical model that displays multiplication as a scaling operation). In the following episodes, we discuss how students explored and interacted with the *Sunrule*.

Episode 1: Sara's initial encounter with the *Sunrule*

One group consisted of two students, Zak and Sara[4]. The second author launched the exploration activity for them by asking, "Any idea what this box does?" Although Sara declared that she did not know what the device did, she oriented the device in the intended way. Figure 4 shows how she positioned the *Sunrule* so that it was aligned with the azimuth of the sun (i.e., the compass heading of the sun, or the place where the sun would appear if it were brought down to the horizon).

This caused the shadows of the gnomons to fall parallel to the strip of rulings on its surface. In that instance, Sara may not have understood the mathematical affordances of the device, but she instinctively positioned it correctly. Then, she changed the device's angle of inclination by tilting the device toward and then away from the sun. This caused the shadows of the gnomons to shorten and then lengthen, respectively, as shown in Figure 5.

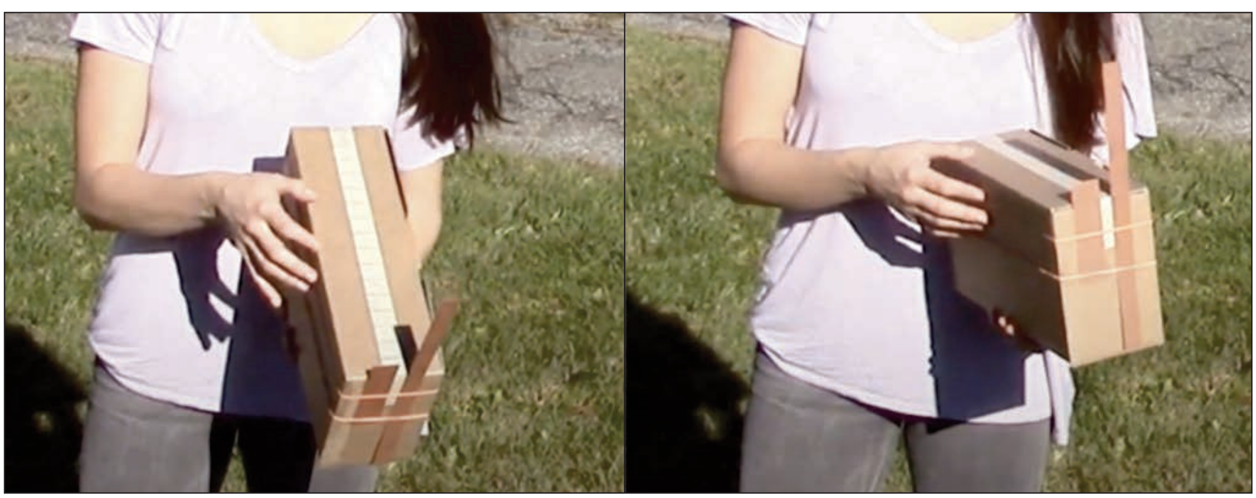
As Sara varied the angle of inclination, she and Zak speculated that the device indicated a relationship between the sun and the shadows. Sara noted the significance of the angle of inclination to the length of the shadows. She said, "It really depends on how you hold it, like, if you tilt it towards [sic] the sun, then the shadows become very short. If you tilt it away from the sun, the shadows get a lot longer." These initial interactions that varied the lengths of the shadows by changing the angle of inclination are the core of the mathematical design of the *Sunrule*. This feature was salient for Sara almost immediately. Her recognition of the significance of tilting the device suggested that the grounding predicate for the geometric definition of multiplication is a natural and potentially powerful embodiment for a continuous scaling conception of multiplication.

Figure 4

Sara orienting the Sunrule *so that the shadows of the gnomons are aligned with the rulings on the shadow plane*

Figure 5

Sara increases the angle of inclination (left frame) and then decreases the angle of inclination (right frame), causing the lengths of the shadows to decrease (left frame) and increase (right frame)



[4] All names are pseudonyms.

Episode 2: Varying the angle of inclination

The second group of students consisted of Sheila, Martha, and Donna. Like Sara, Sheila explored the device by varying its angle of inclination to the sun. After Sheila, Martha, and Donna explored the device for a few minutes, Sheila shared her observation with the group. She said, "When the big one is at 4, and the little one is at 1, so then when you make the big one 8 and the little one 2…they change by the same amount respective to each other." As she said this, Sheila adjusted the angle of the inclination of the device away from the sun. This caused the shadows of the gnomons to lengthen in a movement that was similar to how Sara changed the tilt of the device.

The second author asked the group for the name of the operation demonstrated by varying the lengths of the shadows. Martha replied, "I don't think we know what it is called." Sheila, like Sara, focused on the variability of the angle of inclination as a key affordance of the device. Sheila linked this general observation to specific pairs of numbers (1, 4) and (2, 8) but could not identify the mathematical relationship she had noticed.

After 10 minutes of exploration, both groups had observed that the angle of inclination of the device determines the ratio between the height of a gnomon and the length of its shadow. For example, Sara observed that if the longer gnomon were set to be two or three times the height of the shorter gnomon, that difference in height would be carried through the variations in the lengths of the shadows as the angle of the inclination was changed. The second author assembled the groups in a socially-distanced semicircle because neither group had connected their observations about ratio to multiplication. He summarized the ratio ideas each group had discussed and told them that the device models multiplication. His decision was framed by the reality that this investigation occurred within the context of an elementary methods class. He wanted to ensure that the teacher candidates would recognize the *Sunrule* as a physical, tangible model of multiplication. In future studies with elementary teacher candidates, we would allow more time for open-ended exploration of the device.

Episode 3: Modeling division with the *Sunrule*

In their discussion of multiplication, Zak and Sara realized that the device could also be used to represent division. Zak demonstrated this idea by showing how the multiplication problem $2 \times 5 = 10$ could be interpreted as the division problem $10 \div 5 = 2$. To use the *Sunrule* to divide two numbers, let the height of the multiplicand gnomon be the divisor and then vary the angle of inclination, so the length of its shadow is equal to the dividend. The quotient will then be given by the length of the unit gnomon's shadow. Zak demonstrated this while narrating his *Sunrule* manipulations. Sara and Zak's explorations of the connection between multiplication and division underscore the rich pedagogical opportunities of the *Sunrule*.

Discussion

Conceptual explorations of multiplication

A feature of the *Sunrule* is that its use of movement creates opportunities to differentiate *multiplicands* from *multipliers.* The angle of inclination of the shadow plane determines the multiplier, and the height of the adjustable gnomon determines the multiplicand. Thus, the *Sunrule* creates the possibility of problematizing commutativity because multiplying 2×3, for example, involves different movements than multiplying 3×2. In the former case, the shadow plane is inclined so that the length of the unit shadow is two and the height of the multiplicand gnomon is 3; whereas in the latter case, these movements are reversed. That the product in each instance is six may not be surprising for students, especially if they have had experience with multiplication. But the physical differences between multiplier and multiplicand could suggest questions for discussion: *Why can changes in the angle of inclination of the shadow plane be offset by adjusting the height of the gnomon? Does this work for all products? Why or why not?*

A second activity could help students explore families of products. Questions such as *What are the ways to make 15 by multiplying two numbers?* are fixtures in elementary mathematics classrooms. With the *Sunrule*, such questions could be explored in new ways. In particular, because the *Sunrule* is a continuous model of multiplication, it could lead students to consider not only pairs of whole number factors whose product is a given number but also pairs of rational numbers. For example, from $3 \times 5 = 15$, one could slightly decrease the angle of inclination of the shadow plane to increase the multiplier and then slightly decrease the height of the multiplicand gnomon in such a way that its shadow is still 15 units long. This could lead to pairs of rational factors of 15, like $3\frac{1}{3}$ and $4\frac{1}{2}$, or $3\frac{3}{5}$ and $4\frac{1}{6}$. A challenge would be reading the fractions from the incremental rulings. Still, even if their exact values were difficult to read from the rulings, the existence of such rational factors would be evident from the interplay of the shadows. This could lead to discussions of how many pairs of rational numbers there are whose product is a given number. Such activity could help blur the boundaries between rational and whole numbers (Dimmel & Pandiscio, 2020) and provide an opportunity for students to explore how families of products can be related by continuous variation.

Connections to geometry

The *Sunrule* is like a dynamic diagram in that it allows for the exploration of families of products through continuous movements. Equally significant, the device is a rare example of a mathematical tool optimized for use outdoors. The *Sunrule* harnesses the unique geometry of sunlight to create parallel shadows, the lengths of which are controlled by varying the angle of inclination to the sun to set the multiplier and by adjusting the height of the longer gnomon to set the multiplicand. This report focused on arithmetic descriptions of multiplication, but the *Sunrule* also creates opportunities for exploring multiplication geometrically. Such explorations would be appropriate for secondary students, for whom the *Sunrule* could create opportunities to examine how multiplication, similar triangles, and proportionality are related. For example, one activity for secondary geometry students would be to explore the multiplicative identity. Questions such as, *At what apparent altitude(s) of the sun will the multiplier be 1?* and, *What is the relationship between the apparent altitude of the sun and the magnitude of the multiplier?* could create opportunities for geometry students to probe the trigonometric applications of the device. The *Sunrule* also creates an opportunity for teachers of geometry to celebrate sunlight as the quintessential real-world example of parallel lines. Another series of activities could be grouped as design questions. Examples in this category might be, *Do the gnomons need to be perpendicular to the shadow plane? If the gnomons do not need to be perpendicular, what are the requirements for the position of the gnomons? and, What happens if we change the unit length?*

Limitations

The principal limitations of the *Sunrule* concern its accuracy. We have identified four inherent physical defects that introduce errors in its calculation. By inherent, we mean these defects are a consequence of their physicality—they can be managed but never eliminated. The first source of error is the angle that the gnomons make with the shadow plane. The closer the gnomons are to perpendicular, the greater the accuracy. The second source of error is the flatness or uniformity of the shadow plane. The closer this is to perfectly flat, the greater the accuracy. The third source of error is the resolution, or sharpness, of the shadows. The fourth limitation concerns the accuracy with which the gnomons and the shadow plane are ruled and marked. This constellation of physical errors leads to another limitation: The *Sunrule* can only effectively model a relatively small range of products. For example, the *Sunrules* described in this report had a maximum length of 20 units, which meant that any product greater than 20 would be *off the board.* A possible solution to the limited range problem is to use place value, so *18 × 20* would be off the board, but *1.8 × 2* would not. This is how slide rules were able to multiply a wide range of numbers on relatively small scales. But the reliability of these calculations depends on minimizing the inherent errors.

Although the inherent material flaws affect the accuracy of the multiplication, the overall process of multiplying and the relationship between the multiplier, multiplicand, and product can be explored with the device. The gnomons and shadow planes are easily manipulated to display a range of numerical combinations. We are exploring various designs and production quality choices that would minimize the errors and maximize the range of numbers that can be multiplied. We envision a version where the markings are as precise as a standard school ruler.

Conclusion

The *Sunrule* uses sunlight, an affordance of the world, to model multiplication, a mathematical operation. Simultaneously, it shares a mathematically valid and robust representation of multiplication that is often missing in elementary school classrooms—multiplication as continuous scaling (Dimmel & Pandiscio, 2020; Kosko, 2019). By using a feature of the world to build a mathematical model, the *Sunrule* represents an inversion of what is typically encountered in real-world mathematics.

The COVID-19 pandemic has reconfigured social life. For schools, this has meant adapting instruction to remote, hybrid, or outdoor modalities, among other innovations, some of which may endure even when COVID-19 has been mitigated. The *Sunrule* provides a concrete material context for doing a mathematical activity outside—not simply for the sake of being outside, but because being outside is essential to use the device to do mathematical work. It is a variable, tangible device for modeling families of multiplication problems and probing their mathematical structure. Beyond arithmetical utility, activities with the *Sunrule* could pull students away from screens and create opportunities for students and teachers to reflect on how the geometry of sunlight is integrated with its design. These would be desirable outcomes at any time, and they are especially urgent in the face of the disruptions to teaching and learning brought on by the pandemic.

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Modeling as Story-Building and Storytelling: Developing the Mathematical Identities of Adolescent Girls of Color

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ABSTRACT "Algebra-as-gatekeeper" is a powerful paradigm that structures students' experiences within mathematics class, as well as their future educational trajectories. In this paper, I show how modeling—when conceived as a culturally relevant practice—can support students to make sense of the key ideas of algebra as well as develop positive mathematical identities. This requires two important facets: 1) that students see themselves reflected in the contexts that are mathematized and 2) that they are positioned to be creative, sense-making story builders and storytellers of this mathematics. Results from my semester-long design study with adolescent girls of color illustrate the potential of modeling, especially for students who often experience the greatest barriers to participation, engagement, and learning within the high school mathematics class.

KEYWORDS *modeling, algebra, students of color, culturally relevant pedagogy, storytelling, design study*

For many, algebra is not simply a required high school course but a determiner of students' educational trajectories. Unlike other secondary mathematics courses, it not only mediates access to higher-level secondary mathematics but also serves as a gatekeeper to college. In fact, it has functioned as a gatekeeper for decades (Ladson-Billings, 1998; National Research Council, 1998)—effectively creating a system such that "passing an algebra course becomes a barrier to educational and economic advancement" (Greer, 2008, p. 425). Further, "knowing algebra" can be considered a cultural hallmark of a mathematically literate person; and since literacy and citizenship are profoundly linked in our culture, algebra shapes notions of citizenry as well (Moses & Cobb, 2001).

More troubling, it is well known that students of color—specifically Black and Latinx—and low-income students are more likely to be placed (and kept) within lower-track mathematics courses such as algebra (Cortes et al., 2013). Tracking has been widely criticized for impeding the academic progress of students and exacerbating, rather than minimizing, existing inequalities (Oakes, 2005; Powell et al., 1985). In fact, tracking is considered socially unjust since low-income students and students of color are overrepresented in these "dead-end" courses (Oakes, 2005), where they experience both low-level content and diminished expectations of them as learners (Powell et al., 1985). "Algebra-as-gatekeeper" is a tracking paradigm held together by high-stakes testing. Typically, when students do not pass their high school algebra course—or its affiliated exit exam or end-of-course exam—they are labeled as "repeaters" (a reductive, deficit label) and placed in a variety of intervention models, many of which have already been shown not to be effective (Boaler & Sengupta-Irving, 2016; Cortes et al., 2013; Fong et al., 2014). "Repeaters" courses disproportionally affect students of color, and particularly those from low-income communities, who seem to exist in a sort of educational purgatory I call "endless algebra."

I wanted to understand the experiences of students caught within this paradigm in order to design alternative experiences. Over the course of a semester, I examined the effect of changing the conditions in which the participants came to experience mathematics. I chose an all-girls public high school as the research setting precisely so that I could study, and respond to, the specific issues that adolescent girls of color face within the secondary mathematics classroom related to their agency, their engagement with mathematics, and the intersection of their mathematical identifies with other identities (Joseph et al., 2019; Radovic et al., 2017). Scholars have demonstrated the ways that mathematics socialization and mathematics identity development are critical aspects to the learning and participation of students, particularly Black students (Berry, 2008; Martin, 2007). Because learning mathematics and developing a mathematical identity are inseparable, I pursued two related research questions:

1. How does modeling develop key ideas in algebra, such as functional reasoning?
2. In what ways does modeling support the development of mathematical identities for adolescent girls of color?

In this paper, I describe one important finding from this larger qualitative design study: When mathematical modeling is reconceived as a form of story-building and storytelling, a wider range of learners will a) see themselves within mathematical contexts, b) begin to consider themselves creators of new mathematics, and c) make sense of key concepts within algebra. First, I will define mathematical modeling as it is commonly understood. Then, I will briefly describe the research methodologies used. Next, I will provide evidence of this salient finding. Finally, I will offer a brief discussion of the potential and the limitations of mathematical modeling as an intervention for students like the ones in my study.

What is mathematical modeling?

The relationship between mathematics and the real world lies at the heart of mathematical modeling. That mathematics is a "human endeavor" (Jacobs, 1994), inextricably linked to cultural and social phenomena, is a well-established idea. The term mathematical modeling arose out of the field of mathematics and can be found in almost every facet of science and social science (Pollak, 2011). Often used broadly with conflicting meanings, there is some consensus about two types of central activities to modeling: translating some aspect of the real world into mathematical terms (Gravemeijer, 1997) in order to solve or analyze a situation; and a cyclic process that includes: describing the problem, establishing limiting assumptions, structuring a situation mathematically, selecting and omitting variables, predicting results, interpreting, and validating possible results (Blum, 2002).

Since modeling always originates *outside* of the mathematical domain—a situation, a context—it is often conflated with the solving of "word problems." Learning to solve "real-life" problems is often assumed to be more challenging than solving their "anesthetized counterparts in textbooks and tests" (Lesh & Doerr, 2003, p. 4). But modeling differs from solving word problems in a few important ways: it does not result in "short answers to narrowly specified questions" (Lesh & Doerr, 2003, p. 3) but, rather, a generalizable method that can be shared and reused across related situations; the process tends to be less sanitized, more iterative, and more complex; and more than one mathematically valid model typically emerges.

Methodology: Why a Design Study?

Design studies have been characterized, with varying emphasis, as iterative, process-focused, utility-oriented, and theory-driven (Cobb et al., 2003). Given these key features, design studies afford several advantages: testing theories iteratively and in-the-moment within the context; treating participants as co-constructors of knowledge; tackling everyday dilemmas in classrooms, schools, or systems; recognizing the constraints of theory as it applies to actual participants; and sharpening theory within an educational context. Of course, all scientific research must provide evidence to support its theoretical claims; with design studies, this evidence typically emerges from highly complex and dynamic situations such as classrooms, or what has been called the "crucible of practice" (Shavelson et al., 2003).

Given my desire to study learning and identity development up close and in-depth, a design study, or design experiment, provided the best opportunity to investigate while generating some emergent theories. I situated my study within a high school algebra class in a large, urban, public, all-girls school. After conducting initial classroom observations and in-depth interviews, I selected a "repeaters" course where both the teacher and students were interested in supporting the research. The students—all girls of color—were programmed for this class because they had not passed Algebra I somewhere between one and four times, including summer school.

The class had twenty students enrolled, so I narrowed my gaze in order to focus more deeply on six students who consistently attended class, agreed to be participants, and were representative of the class as a whole. I selected, designed, and modified a sequence of model-eliciting and model-interpreting activities that were designed to support the development of the concept of function while also deepening the participants' mathematical identities. That is, I hoped to engender not only a deeper conceptual understanding of function, a central object of algebra, but also to provide opportunities for the participants to re-engage and re-identify positively with mathematics. Over the semester, I collected and analyzed sets of rich data: video recordings of all class sessions, including small group work, formal interviews and informal interactions with the students, and all student work related to the modeling process.

Whose worlds are being mathematized?

Research consistently describes the ways that students do not see themselves or the "real world" within many school mathematics contexts. Boaler (2008) claims that to do well in mathematics class, children must suspend reality and accept nonsensical problems where, for example, trains travel towards each other on the same tracks, and people paint houses at identical speeds all day long. Further, she has argued that students come to understand that "if they think about the problems and use what they understand from life, then they will fail" (p. 51).

Modeling attends to notions of the "real world," requiring learners to situate mathematical ideas within a realistic or believable world, but the question remains: whose worlds are being mathematized? Often, teachers select contexts that we hope our students will relate to. Yet, the same cultural mismatch that students tend to experience with traditional word problems is not necessarily resolved when modeling. There are a few attempts within the modeling literature to attend to the cultural lives of students when designing and enacting tasks (Cirillo et al., 2016). But research on modeling, as a whole, doesn't concern itself with aspects of students' culture. For this reason, the modeling tasks used in this study were deliberately re-situated in imaginable culturally relevant contexts. This meant studying the participants as girls of color within the classroom, as well as other social spaces within the school setting. In listening, observing, and developing relationships with the participants, I came to understand their interests, passions, beliefs, and lived experiences, which became fodder for contexts to be modeled.

That the participants saw themselves in the mathematics was critical. But it was equally vital that they were treated as both story-makers and storytellers, giving them ongoing opportunities to envision, participate in, and shape the narratives. It was important that the students came to associate storytelling as mathematical in nature—not a diversion (e.g., "off-task" behavior) nor a gimmick (e.g., getting them to do some mathematics), but the vital sense-making work that modeling demanded. As I will demonstrate, this crafting of narratives—personal, cultural, individual, and shared—created conditions under which mathematical identities grew more agentic and more aligned with other salient identities.

With these two ideas in mind—seeing oneself within the contexts and seeing oneself as a storyteller of mathematics—I began to envision modeling as a form of culturally relevant pedagogy (Ladson-Billings, 1995; Gay, 2000). Specifically, I was drawn to story-making and storytelling as a culturally relevant teaching practice. Dyson and Genishi (1994) describe humans' "basic need for story," which they define as a process of "organizing our experiences into tales of important happenings" (p. 2). This idea can also be found in mathematics education research, particularly among those interested in humanizing aspects of mathematics. Su (2020), for example, explains how story functions for math learners, particularly those on the margins:

> Learn a bunch of separate mathematical facts, and it is just a heap of stones. To build a house, you have to know how the stones fit together. That's why memorizing times tables is boring: because they are a heap of stones. But looking for patterns in those tables and understanding why they happen—that's building a house. And house builders perform better in mathematics; data show that the lowest-achieving students in math are those who use memorization techniques, and the highest-achieving students are those who see math as a set of connected big ideas. (p. 38)

Here, he posits that story can serve as a unifying force—the metaphorical "glue"—to hold together and make sense of otherwise abstract, disconnected, or disparate ideas. As such, the idea of mathematics-as-story could be especially powerful for those students in lower-tracked mathematics who were subject to a curriculum of disconnected and unrelated skills and facts and not as a "set of connected big ideas."

In the sections that follow, I will illustrate how I enacted modeling experiences to invite participants to be both story-makers and storytellers, giving them continued opportunities to envision, imagine, and participate in the narratives while developing important ideas within algebra such as function.

Narratives as a form of identity

Drawing upon socio-cultural views of identity—particularly those that recognize the complexity and intersectionality of identities (Martin, 2000; Nasir & Saxe, 2003)—I will demonstrate how the girls' existing identities were shaped and developed within the context of our modeling investigations. First, I will situate our modeling work in the students' own descriptions of their experiences both in school and in math class specifically. Drawing upon data from pre-and post-interviews, I will provide a glimpse into the complex identities that existed for the students.

Mathematical identity and its relationship to story

As I designed the mathematics of the study, I was mindful of the themes that emerged from my initial interview data. For example, eleventh grader Natasha described her favorite classes and why mathematics had never been among them:

Natasha:

> English, you don't really have a right or wrong answer in that class. It's just like…you do your own, like, thoughts—you write down your own thoughts, you read. It's just like… it's not like how math is, like there's one answer and one answer only. In English, there's not really an answer.

The reductive right-wrong binary that Natasha experienced in math class was repeatedly described by the participants, where there was little room for "your own thoughts" outside of an "answer." Later, she expanded upon this distinction:

Natasha:

> Because in English, you can sort of like…project your opinion, like it's not just—you could show more of yourself. Like math class is not like—it's not really like—you're not showing much. It's just about numbers and stuff. It's not like about feelings. It's not about like conflicts. It's not about nothing. Like English is about all that.

Here, Natasha located both her feelings and opinions as *outside of* and *separate from* math class. She did not claim they were incompatible, but that for her, they were not experienced within mathematics. The idea that math class was a place where "you're not showing much" spoke directly to her sense of mathematical identity. Natasha explained why, within her English class, she was positioned to reveal more of who she is and why math class had not afforded her this same opportunity. Mathematics class, in her view, was not a place to "do your own thoughts," "project your opinion," or "show more of yourself." These aspects were salient to her own sense of self. In her fullest critique of math class, she claims that "it's not about nothing." That is, mathematics is about nothing that is relevant, important, or meaningful to her.

Tenth grader Danya expressed a different form of disconnectedness from mathematics, noting that she rarely felt successful as a math learner. She described how she currently experienced mathematics teaching and learning:

Danya:

> The way they do it here, it's all math steps, so like that's what makes it more confusing. It was like we did a whole bunch of steps, and it confuses us.

Her conception of mathematics as "all math steps" was indicative of the emphasis within testing environments on procedures (usually at the expense of reasoning, justifying, or developing conceptual ideas) and described by each of the participants in the study. It was consistent with Williams and Miner's (2012) notion that "standardized tests also come packaged with demands for more standardized curriculum. These calls are part of a broader effort to promote a narrow version of what children should learn" (p. 10). The narrowing of mathematics that Danya alluded to is in direct contrast to the expansive work of modeling. As Gann et al. (2016) explain, "When students are engaged in modeling, they are not using an algorithm or following a prescribed set of steps" (p. 102).

In describing her teachers, Danya reinforced the fact that identities are intersectional and constructed socially. She explained how the intersection of race, ethnicity, and gender shaped who she is and how she learns. Most of her teachers, particularly her elementary teachers, were "just so confusing." One middle school teacher, however, was noticeably different:

Danya:

> Then in seventh grade, that's when it was just like, "This is easy." Like, "I know this." I don't know. I felt

> like there was a connecting or something, the way they taught it. Because, like, I'm not trying to be racist, but my teachers have been like American or something, and my seventh-grade teacher, she's actually from Jamaica. So I'm also from the Caribbean, so I don't know if there was like some type of relationship…I just understood the way she taught it.

Learning from an Afro-Caribbean woman helped Danya to make sense, to see math as "easy," and to feel as if "I know this." Instead of "steps," she spoke of understanding, a noticeable shift. She suggested that what made this learning possible was "some type of relationship"—that this personal connection was critical to her understanding of mathematics. The importance of an emotional bond between teachers and their students—to motivate, to engage, to generate learning—is well-established (Waller, 1932; Lee et al., 1997). In this case, the bond between Danya and her teacher was grounded in their shared racialized and gendered identities—a vital connection between them, even if it was never articulated outright.

For Danya, this relationship with her mathematics teacher also re-oriented her towards mathematics and led her to consider her mathematical identity related to, not separate from, other intersecting identities. Research confirms that racial identities develop early (Tatum, 1997), yet Danya's middle school experiences also bolster what Noguera (2003) has noted: "Adolescence is a period when young people begin to solidify their understanding of their racial identities" (p. 20). Understanding how Danya came to see herself as an Afro-Caribbean girl has a significant bearing on the development of her mathematical identity, as these are dynamic and intersectional. Given the associations of mathematics (and for that matter, all things considered as "the norm") with whiteness, maleness, and middle-class status (Rubel, 2019), it was vital that Danya saw herself, and others like her, within the community and practice of mathematics.

Because of these early analyses, it was critical that the modeling investigations evolved from imaginable, culturally relevant stories so that each of the girls felt some personal connection and interest. Typically, this happened either by introducing an artifact (e.g., an object, an image, a video) or by orchestrating some common referent (e.g., a shared experience). It was necessary, but not always sufficient, simply to situate mathematics in contexts that might resonate culturally with the students. After all, there is a well-established tradition of "dressing up" mathematics in contexts to engage or inspire learners, often with limited success. What was equally important was to elevate the students as story-makers and storytellers as well—that is, to give them a specific role in creating the mathematics through story.

Of course, storytelling can be a remarkably intimate act, full of the possibility of connecting with others. The desire to connect with others in school and beyond has been found to be particularly important for adolescent girls (Belenky et al., 1986; Gilligan, 1990). These salient aspects of stories are bolstered by Gay (2000):

> Stories are means for individuals to project and present themselves, declare what is important and valuable, give structure to perceptions, make general facts more meaningful to specific personal lives, connect the self with others, proclaim the self as a cultural being, develop a healthy sense of self, forge new meaning and relationships, or build community. (p. 3)

The use of storytelling in this study was situated within a gendered and racialized context, consistent with what Joseph et al. (2019) described as "transformative pedagogical models" for Black girls within the high school classroom. They proposed that when students were encouraged to "personalize mathematics with examples from their own communities and histories" (p. 138) and given meaningful ways to co-create knowledge, they were more likely to be repositioned within the mathematics classroom. Aguirre et al. (2013) extended this idea, situating it within mathematics:

> Mathematical identities can be expressed in story form. These stories reflect not only what we say and believe about ourselves as mathematical learners but also how others see us in relation to mathematics. (p. 14)

Therefore, the relationship between storytelling and identity-building was critical to the study: we construct and tell stories about ourselves to reveal to others who we are and to become members of communities who share salient aspects of our identities. Within the mathematical context of this study, story and story-building were vital in several ways: allowing students to locate themselves within the contexts of the mathematics; illuminating mathematical identities and connecting to other intersecting identities; and offering new, meaningful roles to students in math class who experienced barriers to learning or participation.

Using artifacts as stories: Determining what is real

In the first modeling task, the girls were invited to study and mathematize a high school graduation program shown in Figure 1. First, I invited them to share their own stories of graduation. A wide-reaching conversation ensued, as nearly every student sharing a detailed story relating to a graduation ceremony they experienced. They described "gap and gown," "diplomas," and the ritual of family and friend photos. The stories transitioned across time and space, from fifth-grade graduations to imagined college graduations. For some, like Keisha, the ceremony was a significant feature: "In the graduation, my parents are going to cry. They take photos and more photos. My mother cries through everything." For others, like Jaila, it was the meaning of the graduation that mattered. "The only thing I care about," she explained, "is getting my diploma. I don't care about the graduation. I just need to know I'm not in high school anymore." When storytelling is connected to larger pedagogical goals, this small example of personalizing the mathematics shows how it is both appropriate and useful within the context of teaching and learning.

Once the idea of graduation was connected to personal experiences, I introduced the artifact. Tenth grader Danya noted publicly, "I have a lot of questions. Now let me see if I can find someone with my name." This inspired her classmates to look for their own names as well:

Keisha:
Oh my gosh, there was a Keisha there. Natasha, I found you.

Jaila:
There's no Jaila.

Danya:
There's no Khadijah.

Keisha:
There's a Kristin. Do you want a Kristin?

Figure 1

High School Graduation Program

San Lorenzo Valley High School
Class of 2011

Micah Stephen Adams
Xavier Josue Alvin
Connor James Antisdel
David Zachary Baker
Bradley Garrett Barnard
Joseph Barnes
James Alexander Barnes
Michael Barton
Sarah Raisha Beasley
Allison Rose Beasley
Emily Anne Bechtle
Brandon Scott Beevers
Patrick Fereva Belardi
Kaitlyn Ella Benson
Jordan Starr Thomas Bereman
Taylor Renee Foster Berritto
Ashley Michelle Ann Billington
Morgen King Biswa
Jaclyn Kate Black
Xitlali Borreson
Chere Nicole Brandon
Sydney Lauren Garcia Breili
Curtis Martin Brewer
Katherine Evelyn Brown
Annika Bruce
Amanda Lynn Bruce
David Michael Burgler
Jourdan Donovan Burk
Nicholas Ryan Burks
Joseph Thomas Burton
McLean Avery Camacho
Alexander Austin Campbell
Jason William McGregor
Taylor James Casey
Kory Daniel Chadwick
Audrey Elizabeth Chapin

1:28pm

Amber Anne Chase
Faye Elise Chavez
~~David Thomas Chesus~~
Zoe Darlene Chinn
Emily Elizabeth Clements
Claire Elizabeth Cloud
Steven Dennis Connelly
Conor Thomas D'Amato
Cassandra Suzanne Davis
Lauren Marie Dempewolf
Blake Bradley Dennis
Jake Madison Debreuil
Shannon Elaine Eisner
Emily Anne Engel
Samantha Marie Ferguson
Dominique Angelina Jones
Cherina Marie Freitas
Anju Friend
Nathaniel Justin Fruzza
Mariah Nichole Galmez
Robin Lavendar Garcia
Ronja Andrea Franklin
Mason Garrett
Kayle Anne Genis

1:33pm

Melanie Lauren Gleim
Andrea Serena Godbout
Angela Gonzales
~~[illegible] Goodreau~~
Dakota Makua Gorman
[illegible] Greer
Michael Marie Grindy
Erin Cassidy Groswird
Krista Marie Grunberger
Tyler Alan Hagen
Hazel Gladys Jane Haikkila
Marcus Taylor Halversen

Nicholas Gregory Hargraves
Alexander Samuel Harnett
David Michael Hart
Melissa Lorraine Harvey
Olivia Louise Herrera
Kasia Adair Hill
Ludmila Eliana Hipsley
Jazelle Gina Hooper
Mariah Donna May Hopkins
Lindsey Marie Hoppin
Bryn Kathryn Horton
Thomas John Housek
Ellen Joanne Innis
Jordan Isaacson
Joseph Clifford Jansen
Allison Kirstine Janus
Nettie Pearl Mitchell Johnson
Lacey Marie Johnson
Felipe Johnson
Shelby Danielle Johnson
Zachary David Johnson
Dakota Shea Jones
Tanner William Jones
Clara Joy Kamau
Mia Anne Kellogg
Maxine Elaine Kelly
Ryan Scott Kennedy
Jenny Rae Kersten
Andrew Benjamin King
Russell Theodore Klair
Zachariah Waya Tsgili Klus
Christina Rose Knoll
Jessica Danielle Kraft
Jessica Ann Lacy
Joseph Allan Landry
Abbie Mae Leveque

Matthew Alexander Lipperd
Madeline Elizabeth Lopez
Emily Marie Madison
Rebecca Lynn Makita
Savanna San Mangelsen
Zane Michael Markham
Alfredo Martinez
Tyler Cameron Mattson
Wayne Thomas McCarthy
Theos Christopher McClish
Annelise Marie McFarland
Alexander Michael McIntosh
Dominique Sharee McIntosh
Victoria McKenzie
Violett Josette McNally
Kendra Dee Messimer
Marisa Grace Brudnick
Sage Nicole Monack
Thomas Ocean Moreno
Rainbow Roxanne Muchamuel
Nicholas Jefferey Newberry
George Austin Norfleet
Travis James Nugent
Joseph Carl Olson
Lacie Marie Orlando
Mary Elaine Ivy Orr
Austin Leonard Overton
Haley Lauren Pace
James Evan Paolini
Timothy William Parker
Carson Taylor Paynter
Zachary Alan Peabody
Katrina May Pearce
Samuel Jackson Pelphrey
Chandler Elizabeth Perazzo
Sebastian Thomas Peterson

SOURCE: ILLUSTRATIVE MATHEMATICS

The episode illustrates three important ideas. First, the students were individually and collectively looking for themselves, illustrative of the need to verify the artifact's authenticity or believability. Alternatively, the search for one's name might have been symbolic of students' desire to be "seen" within the mathematics classroom. That is, finding one's name—particularly if it was unique or culturally specific—validated that this was not the teacher's mathematics but, in fact, a part of students' own mathematics. Second, the girls were beginning to make their "noticings" public and shared. Noticing, as a valued mathematical behavior, was a new idea and allowed them to move beyond the "right-wrong" paradigm of mathematics. There was no risk of being "wrong" because "wrong" did not exist within this context. As such, noticing provided all participants a way into the conversation, regardless of their perceived status. Third, this phase of modeling, *problem posing,* led to the codifying of the students' questions. Those questions, in turn, would later structure how they mathematized the situation. But, the questions needed to be grounded in the context, and therefore the students needed to examine and make sense of the program as they were doing here.

Debra changed the direction of the conversation entirely when she declared, "These names don't seem real." Encouraged to explain, she said, "These names are foreign." She began to read aloud a few names to justify her belief: *Ringelman, Rinker, Peabody.* The students burst into laughter at the absurdity of these names. I paused to acknowledge that "these names don't feel like real people to you," and many agreed. In fact, this moment illustrated the importance of culturally relevant artifacts as springboards for learning. Debra, who identifies as African American, was asserting that these White-sounding names fell outside of her experience and world. Her use of the word "foreign" reinforced the strangeness of these names for her. She named the cultural mismatch between this artifact and her lived experiences. Our conversation about the believability of the program, and the students' skepticism about it, shifted into something broader:

Danya:

What is this supposed to show us?

Jaila:

Well, what relation does this have with math?

I reminded everyone of the graduation stories we had just shared and encouraged them to think of this graduation, strange names and all, as one we could imagine together. My efforts were grounded in ideas about story: We began with real stories from the students' lives, transitioned into a context *not believable/imaginable* to the students. To reconcile this mismatch, we were now in a new space of stories that could be *imagined.* As a white teacher-researcher, I knew that my chosen artifact had failed to connect with the students' lives and, in fact, had the potential to further alienate them from the mathematics. Yet, their critique of the "realness" of the artifact was consistent with initial interview data, in which the girls articulated a desire to be seen within mathematics and to create mathematics related to their own lives.

Beyond the artifact: The telling and re-telling of high school graduation

Once in small groups, the students investigated a question they had generated together in the problem-posing phase of modeling. For many, this meant beginning to model the situation of the graduation ceremony and, later, to build a function of it. Keisha, Jaila, and Danya wondered when the graduation program began. They used mathematical details in the program to develop a generalizable rate of five seconds per name. When Jaila doubted this rate, her partners re-told the name-calling portion of the graduation, using what Keisha and Danya called its "regular beat" and "correct" tempo to attempt to convince her that this mathematical claim was reasonable. Not yet convinced, Jaila asked them to "do it one more time," this time noting that "you sound like the people," presumably the teachers or administrators calling graduates to the stage. This exchange, where mathematics was performed, not written, was one of many throughout the study and was consistent with Hammond's (2015) theories about culturally relevant teaching within African American traditions. "By telling stories and coding knowledge into songs, chants, proverbs, and poetry," she wrote, "groups with a strong oral tradition record and sustain their cultures and cultural identities by word of mouth" (p. 15). That Danya, who identified as Afro-Caribbean, and Keisha, who identified as African American, engaged in modeling as a form of performance or re-enactment, supported them to be active, creative, confident participants within their math class.

Later, the students revised their rate to five names per minute or twelve seconds per name. Wanting to know how this revision occurred, I probed further:

Miss Imm:

Does that seem right?

Danya:

Yeah.

Keisha:
Yep.

Miss Imm:
Do you think you could convince the group that it's twelve seconds per name?

To do so, Keisha embellished the story of the graduation:

Keisha:
She's in heels. That's right 'cause when you're in heels…And then most auditoriums are slanted. So, you have your heels, and it's kind of….[gesturing a slight slope with her hands]

Keisha also wondered whether they call your "full name." When Danya confirmed that they do, they searched for long names in the program (e.g., Netty Pearl Mitchell Johnson) to illustrate that these students, coupled with walking in heels on a slanted surface, might really take twelve seconds to cross the stage.

Taken together, these interactions demonstrate how story building and storytelling provide opportunities for students to make sense of mathematics from their lived knowledge. The "graduation" task was chosen to launch the study to uncover participants' prior knowledge and to connect that knowledge with new algebraic concepts. But more importantly, it gave the students a vivid experience of mathematics that was different than any they experienced before. The iterative "dance" of modeling between the context and the mathematics through story—the way they envisioned, reenacted, critiqued, embellished, and invented—allowed each of them to co-construct this mathematics with meaning. Because the central activity was modeling—where the context cannot be trivialized or separated from the mathematics—this narrative illustrates a critical feature of what it means to do mathematics. As illustrated previously, each time the girls returned to the story of the graduation, they used both the narrative *and* mathematical details to test, express or revise their model. As one example, Keisha adds her own details to the story that are not expressed in the artifact (e.g., heels, slanted auditorium) to justify the rate. This need for congruence between mathematical and lived worlds is at the heart of modeling.

Modeling as a collaborative process, not a singular product

A few days later, one student per group was chosen to defend their models to the class to discuss and verify them together. Each group made a poster of their thinking, displayed prominently as they presented. With her partner absent, Khadijah was asked to represent her group.

Khadijah:
So, I'm by myself?

Miss Imm:
Do you feel like you could represent your thinking? What's on the poster?

Khadijah:
I'm pretty sure I could do it… I'm pretty sure…

Then, tapping on the various ratio tables on her poster, she notes, "I've probably forgotten one thing, but I think I've got the process, so I should be fine." That the model on her poster is a result of a process is further evidence that modeling supported students like Khadijah to deepen her ideas about rate. When she first began modeling, she, like her peers, over-relied on social knowledge—shared or collective ideas developed in relationship to others—satisfied by anything that "felt" or "seemed" reasonable and not interested in justifying her instincts with mathematics. Here, however, social knowledge played a different role in her sense-making; she used it to bolster, not replace, mathematical claims. She trusted the unit rate that she and Natasha constructed could explain how it was equivalent to other rates in the graduation program and began to connect rate to the idea of function by identifying the two variables (*names called* and *minutes*) and explaining their linearity on a graph.

Modeling across these episodes provided Khadijah a chance to build on her emergent ideas about rate in three significant ways. First, that the mathematics of the context made sense to her allowed her to continually check her understanding against the "common sense" of the story. Each time she expressed hesitancy about the mathematics, she returned to the context to justify her ideas. Second, because she was not problem solving independently but modeling in a partnership, her thinking about rate was co-constructed. Third, because modeling was iterative and included a public defense, Khadijah's ideas (and confidence about those ideas) deepened over the study. She had multiple opportunities over several weeks to construct (and reconstruct) ideas about rate, ratio, equivalence, and function. She came to appreciate that our goals had shifted: from getting the right answer (e.g., *product*) to modeling as a generalizable *process,* as illustrated by her statement, "I think I've got the process, so I should be fine."

Several weeks later, in a new modeling task, Khadijah was again asked to present to the class on behalf of her partner. She noted that many students were absent:

Khadijah:
I think we should wait 'til everyone else comes because everyone else has to be here, too, as a class, to know.

Miss Murray:
We can wait if you want.

Khadijah:
Because we have to have the whole class here, too, so they would understand it.

The episode illustrates an important turn—that participants were now producing mathematics for each other and for the class. As such, it was important that "everyone else"—the "whole class"—was present. The subtlety of their language is critical in demonstrating how much their view shifted. Though their teacher referred to the phase of modeling as "present your information"—potentially a one-way transmission, not a discussion—Khadijah understood that the class must be present "so they would understand it." In framing the presentation as a chance for her peers to discuss and make sense of their thinking, she went beyond a one-way presentation to her peers. She knew that she had to convince the class of the viability of their model and be willing to defend it when her peers scrutinized it. This is consistent with Anderson (2007), who noted the importance of the collective: "Learning mathematics involves the development of each student's identity as a member of the mathematics classroom community" (p. 7). Collaborative activity, such as modeling, has particular value among adolescent girls who tend to enjoy and become skilled at working together, learning to trust one other, and ultimately elevating each other's voices within the classroom space (Kuriloff et al., 2017). Studies confirm that modeling is best enacted as a group activity (Ikeda et al., 2007) where discussions elevate the collective understanding.

This exchange also suggests that modeling helped to reposition the girls relative to mathematical knowledge. Specifically, it suggests the ways that modeling can support students to be *creators* and not simply *receivers* or *reproducers* of mathematical ideas. Boaler (2002) found that students, particularly in traditional classrooms, were often given a limited way of participating within math class— what Belenky et al. referred to as "received knowing" (1986, p. 4). This was the result of having mathematical ideas presented to them by teachers, or textbooks, as the girls described throughout the study. When they were modeling, as I have shown, the girls were neither receiving nor reproducing mathematical knowledge; they were creating it themselves. This supported them to interpret ideas, make choices together, exercise their own thoughts, and exhibit agency, all of which contributed to more integrated and positive mathematical identities.

The importance of explanatory narratives: Cell phone task

Later in the study, the students were given a set of cell phone images shown in Figure 2 (next page) and invited to notice, name, and wonder. The task emerged from my observations of their early-afternoon scurry to find empty classroom outlets to charge their phones before dismissal. The task was designed to a) provide deeper opportunities to model and make sense of function, b) build upon a well-known cultural object (e.g., iPhone), and c) allow the students to generate interpretative stories as part of the modeling process.

When we first introduced the cell phone images, we were careful to say little about the owner of the phone. Yet, almost instantly, the students had gendered the cell phone owner as male, based largely on his perceived importance:

Natasha:
Does he have a job, like a serious job? That's probably why he has so many emails.

Miss Murray:
Most definitely has a serious job.

Sam:
Is he like a businessman or like a lawyer or like —

Miss Imm:
More serious than that.

Khadijah:
A CEO or something?

Jaila:
He works for the government?

Danya:
The President?

Only a busy or important person (e.g., man), they reasoned, would have 31,000 emails. This association of important people with maleness provided a window into how gender had shaped their own identities as girls of color. Within the context of mathematics education, this is consistent with other analyses such as Mednick's (2005) study of how the construct of being "good at math" was shaped by gender and, as a result, taken up less frequently by girls. Further, Rodd & Bartholomew (2006) noted how participation within college mathematics was complex for young women, who experienced both invisibility and "specialness" related to their participation.

After realizing no more "character details" would be provided, the students moved towards understanding the users' behavior and usage patterns. They became

Figure 1

Cell Phone Images

COURTESY OF THE AUTHOR

focused on determining why the battery was declining unevenly over two intervals, which later drove their mathematical modeling. Natasha first described the difference:

Natasha:

I was saying—because from 4:52 to 5:06 is only 14 minutes, and his phone went down 4 percent. So I say he was on his phone, 'cause from 5:51 to 7:13 his phone only went down 5 percent, and that's a longer, like, time distance there. You know what I'm trying to say? Yeah.

Knowing that the rate of battery decline was not steady across the intervals (i.e., a nonlinear function), the students began to develop explanatory narratives about *why* this difference existed:

Keisha:

It's probably…you know what I feel like? And I think people who have iPhones would agree—it was probably unlocked, and the screen was on, but he just wasn't doing anything to actually kill the battery as quick as he did the first.

Jaila:

Mm-hmmm, 'cause grownups, I see them on the train. They don't lock their screen! I'm just sitting there like… "Can you lock your phone?" [throws hands up, eyes widen]

Keisha:

[laughing] Yeah, like they'll sit there and with their phone unlocked, like, brightness up and everything.

The exchange was important for several reasons. First, when Keisha first proposed an explanatory narrative, she treated her peers as fellow iPhone owners, noting that her expertise was shared since "we all have iPhones...we *know* them." Second, Jaila bolstered Keisha's argument with observations from her daily commute—a lived experience that gave her credible knowledge about how adults behave on their phones. Her exasperation at adults' refusal to lock their phones generated laughter among the group. Unlike kids, Jaila claimed, these "grownups" simply did not understand how to use their phones. She demonstrated *how* her knowledge of iPhones was deeper than that of the adults around her. Third, there was no teacher facilitating or structuring this dialogue, and no need for an adult to interrupt or guide this conversation. The artifact of the cell phone images alone served as a springboard for authentic, imaginative storytelling. Because of my insistence that modeling was a collaborative endeavor and that we make mathematics for each other (not the teacher), stories such as these were co-constructed—it was the students' narrative to craft and later justify.

To a skeptic, episodes like these might suggest students were "off task"—further from the mathematical goals of the algebra course, but this is where modeling is uniquely positioned to be cast as story-building and storytelling mathematics. There can be no mathematics in modeling without an imaginable, full-of-commonsense context in which to situate the ideas. Unless students can really immerse themselves in a situation—to personalize it and play an active role in the construction of meaning—then these contexts are just as flimsy (and useless) as the ones they had experienced before.

Discussion and implications

Stories reflect not only what we believe about ourselves as math learners but also how others may see us in relation to mathematics. Many influences—teachers, families, peers, schooling, and testing—play a role in the shaping of these identities over time. Mathematics identities are always co-constructed with racial, gender, and class identities. In this paper, I situated the analyses in Natasha's and Danya's experiences of math class—consistent with many pieces of data that suggested that the participants were both disengaged and dis-identified with school mathematics. I posited that modeling could provide an alternative under two important conditions: that each modeling cycle would center around a culturally relevant artifact or shared embodied experience and that students took up these artifacts and experiences as opportunities to create and tell stories. This linking of story-making with mathematics was consistent with the literature and particularly relevant for students who have never experienced mathematics as related, personal, or sensible. According to Su (2020):

> A story creates a narrative from disparate events and connects listeners to itself and to one another. It is not different with mathematics. Connecting ideas is essential for building meaning in mathematics, and those who do it become natural story builders and storytellers. (p. 39)

As the analyses here show, re-framing modeling as mathematical story-making and storytelling positioned the students to take up new roles, and by extension, to deepen their mathematical identities.

Modeling, no matter how broadly conceived, is not a panacea and promises neither a more inclusive experience nor better outcomes for students. It cannot disrupt the large inequitable systems (e.g., tracking, high-stakes testing, systemic racism) that shape students' access to high-quality mathematics. Yet, when modeling is reconceived as being designed for and with students, who will become the story builders and tellers of mathematics, it has the potential to disrupt existing inequitable patterns within classrooms. As such, this study suggests that a vital (but under-theorized) relationship between mathematical modeling, culturally relevant pedagogy, and identity development exists and is worthy of continued study.

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Gerrymandering in the High School Geometry Classroom

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ABSTRACT Teaching gerrymandering in our high school geometry classrooms provides students with a unique opportunity to use mathematics to describe, analyze and make sense of the world around them. Our purpose is to provide our students the opportunity to apply learned definitions and formulas of area and perimeter to a sociopolitical context. We present a unit that we designed for teaching high school students about gerrymandering and describe how teachers may implement this unit in their classrooms. In this unit, students discovered the mechanics of gerrymandering. They made calculations and observations about area, perimeter, and the ratio of area to perimeter. They considered compactness as a potential tool to indicate gerrymandering. They used proportional reasoning to measure the fairness of the partisan split for a given map. We provide supporting examples of student work and discourse and make recommendations for future iterations of this unit.

KEYWORDS *geometry, gerrymandering, equity, high school, projects, apportionment, proportional reasoning, compactness*

Background

Throughout American democracy, gerrymandering has been perpetrated by all political parties and at all levels, from congressional districts to school districts (American Civil Liberties Union [ACLU], 2016). In most states, the legislature draws district boundaries and thus can manipulate them to gain legislative control. There are several states in which nonpartisan, bipartisan, or citizen commissions are used when redistricting. That number has grown in the past several decades (NCSL, 2020). In fact, there are over 200 proposals for redistricting reform being discussed by state legislatures for the 2021 redistricting process (Brennan Center, 2020). These proposals are motivated by an upswing in disproportionate representation after the 2010 census. Republicans gained 63 congressional seats and control of the United States House of Representatives. According to Wines (2019), the Republican party "had poured money and expertise into state legislative races with the specific aim of gaining control over redistricting; the Democratic Party had not" (para. 9). Partisan gerrymandering became a more prevalent part of the political news conversations, with cases in Pennsylvania, Wisconsin, and Maryland being elevated to the Supreme Court (Brenner, 2018). In an amicus brief for two of these cases, Charles et al. (2019) argued that "The Constitution protects an individual's right to an undiluted vote. The government acts unconstitutionally when it intentionally dilutes an individual's vote" (p. 7). The potential dilution of our students' votes and voices is at the core of our motivation to teach gerrymandering in the high school math classroom. We teach high school students from the South Bronx who will be eligible to vote within three years. When surveyed, our students shared that they do not feel the government is connected to them or is for them. We want our students to be informed about the redistricting process and the potential for disproportionate political power. This way, they can hold their government accountable to represent their interests fairly.

Our Schools

Our schools are located in Districts 10 and 12 in the Bronx. Neither school requires testing for admission, and both are public schools. Neither school tracks classes by ability, and we are both integrated co-teachers for about half of our total classes. Students in both schools are roughly two-thirds Hispanic and one-third Black, as reported in the data for our schools. About 90% of students from both schools are eligible for free lunch. While the mission and vision of our schools are quite different, both schools are designed to meet the needs of our students and create a supportive community where students feel respected and seen. They are both small schools of 350–450 students. Since the 1990s, New York City has closed many large comprehensive high schools to create small schools where students are known well, that value smaller class sizes, or that organize around a specific discipline or theme. Bronx Theatre High School is modeled after a theatre company. Students learn to act as an ensemble and to support one another. By taking on a role, they learn empathy and critical thinking. Wherever possible, theatre is integrated into classrooms. For example, in the gerrymandering unit, students take on roles of politicians to debate competing map ideas. Fannie Lou Hamer Freedom High School is part of the New York State Performance Standards Consortium. Rather than standardized testing, students complete a portfolio of project-based assessments for graduation. Many of the student quotations in this article are from papers written by Fannie Lou students.

Unit Outline

This unit builds on the concepts of area, perimeter, and proportional reasoning. Many of our students enter the classroom able to recite a definition for area or perimeter and substitute values into a formula. Our purpose is to provide an opportunity for high school geometry students to apply what they already know about these concepts to a sociopolitical issue. As stated in the Common Core Math Standards for High School Geometric Measurement, Dimension and Modeling, our students use properties of geometric shapes to describe objects and "apply geometric methods to solve design problems" (Council of Chief State School Officers, 2021, para. 1). We spent 4–6 weeks on this unit, but there are many opportunities for extensions that could extend the length at the teacher's discretion.

The unit consists of four parts. In *Contextualization,* we look at the population of each state, the number of seats in the United States House of Representatives for each state, and the partisan split. We ask questions about fairness in terms of how many representatives a state should have given its population and how many we expect should go to Democrats, and how many to Republicans. In *Play the Game,* students play a board game in which they compete to win districts. By playing the game, our objective is for the students to identify the number of votes needed for them to win a district and for them to practice minimizing their own spent votes while maximizing the spending of opponents' votes.

The third part of the unit is centered around *Squaretopia,* a state that gets divided into ten districts. Students learn that the number of districts won by each party entirely depends on how the districts are drawn. Here our goals are for students to draw districts that are contiguous and to identify and use the minimum number of votes needed to win a district in order to maximize the number of districts they can claim.

Finally, in *Calculating and Analyzing Compactness Metrics,* students use four compactness metrics to reason about the gerrymandered Squaretopia maps. Students come to the realization that gerrymandered maps almost never result in compact shapes. This leads to the development and application of formulas that measure compactness. Because there is no single definition of compactness, students practice how to make a mathematical argument using a given metric. Throughout these four parts, students see that when they attempt to make a party win as many districts as possible, that they will likely create "weird" shapes. In the final part, they consider these strange shapes through the lens of compactness and question if the physical shape of a district indicates that gerrymandering has occurred.

Part One: Contextualization

In Appendix A, we provide questions to help students build foundational knowledge of gerrymandering. If students have already taken a course in American History or Government, they may be able to answer questions about the House of Representatives, Senate, and apportionment. If not, we suggest assigning homework as a Google form or handout with the provided guiding questions. Alternatively, the teacher may choose to give a one-period presentation with context about the branches of government and apportionment of congressional representation.

Data from the 2020 Census will inform state apportionment counts and district maps for ten years. Deadlines for finalizing these maps are determined at the state level. Many of those timelines will need to be rushed or altered given COVID-19-related delays in collecting 2020 Census data (NCSL, 2021). We hope that as a result of engaging in this unit, students will be able to watch or engage in the redistricting process with a critical eye.

Part Two: Play the Game

About the Game

Mapmaker: The Gerrymandering Game, shown in Figure 1, is a board game developed in 2018 by three siblings from the Lafair family who live in a gerrymandered district in Austin, TX. Each student plays as one of 4 political parties.

The learning goals are for students to identify the number of votes needed for them to win a district and for students to think critically about simultaneously spending their opponent's votes while minimizing the use of their own. Students strategize about where to place their boundary pieces to win as many districts as possible for their party. They discover the four methods of gerrymandering outlined below and make observations about proportionate and disproportionate representation. Students play this game in groups of 3 or 4, either for one or two 45 minute periods. After students play the game, teachers may help students analyze the game collectively by taking pictures of each completed game board and analyzing the shapes and votes per district with the class.

Figure 1

Mapmaker Game Board (Lafair, 2018)

The *Mapmaker* board is divided into 74 hexagons. There are 40 circular chips in 4 colors to represent four political parties. Before the game begins, randomly place equal numbers of circular chips on the hexagons. Each color has ten chips numbered 1-10 to represent the number of voters who reside in that geographical hexagon. The players take turns drawing boundaries along the existing sides of hexagons to win the most districts for their party. Districts can have between 4 and 7 hexagons. The player with the greatest number of votes within the boundary wins the district. As a scaffold before playing MapMaker, teachers may select several online (e.g., gametheorytest.com/gerry/) or paper options with small arrays of squares in two colors.

Playing the Game in the Geometry Classroom

While the game itself is designed for each player to fight for their party, we recommend that students play the game a few times with varying objectives. In one iteration, players try to make their party win the most districts. In another, all players attempt to maximize the districts for one party. By playing this game, students will discover the following four ways that a party's power can be diluted.

- *Packing* is defined as pushing all of the opposite parties' votes into one district, thus minimizing the number of districts they win.
- *Cracking* is slicing up opponent votes into districts already won by your party, thus rendering those votes meaningless. (Daryl et al., 2020) Essential to the cracking strategy is the identification of a bare minimum number of votes your party needs to claim the district. Once those votes are used, no further votes for your party should be spent in that district.
- *Kidnapping* is when boundaries are drawn to move the home of an incumbent into another district where they are unlikely to win.
- *Hijacking* or double-bunking occurs when boundaries are drawn to place two incumbents in the same district so that one must lose, thus losing a seat for that party (Caltech, 2008).

These four practices are illustrated in Figure 2 (next page). For a helpful audiovisual tool that demonstrates these four concepts, see Propublica's *The Redistricting Song* on YouTube (2011). The teacher may walk around

Figure 2

Four Gerrymandering Strategies

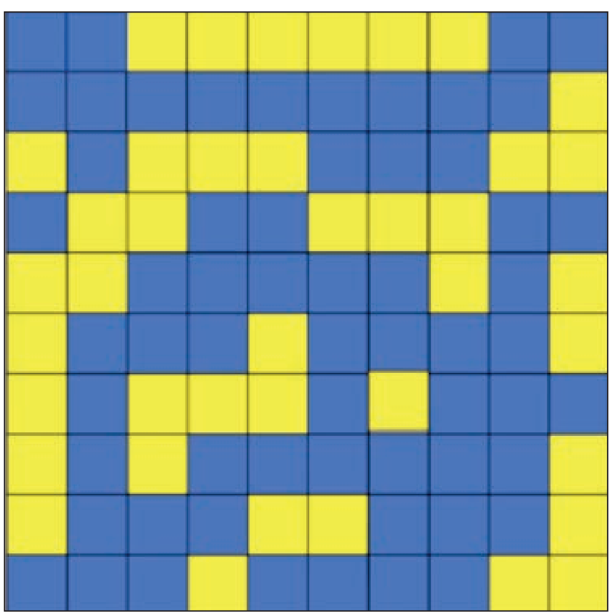
to identify instances where students are implementing these strategies. After the game, invite students to share their strategies and then name them.

Part Three: Squaretopia

This activity was created by the Metric Geometry and Gerrymandering Group (MGGG, 2017). Figure 3 shows a ten-by-ten grid that represents a fictional state, Squaretopia. Imagine a state with 100 voters, 60 voting for the blue party and 40 voting for the yellow party. The legislature of this state decides how to create ten districts with equal numbers of voters per district. In our classrooms, students create a district map for each of the four criteria.

Figure 3

Squaretopia Grid (MGGG, 2017)



1. Create ten compact districts. We define "compact" in this context as the district being as closely packed as possible, minimizing the distance across in any direction. Compactness is discussed in depth in Part Four.
2. Create ten proportional districts. This means to create districts such that the number of districts won by each party is proportional to that party's share of the general vote.
3. Gerrymander for the blue party. There are 60 blue squares in Squaretopia. Each district could theoretically have six blue votes, assuming the physical constraints allow. Try to make the blue party win as many districts as possible.
4. Gerrymander for the yellow party. The yellow party gets 40 votes, so it could win six districts with at least six yellow votes in each of those districts. The learning goals for Squaretopia are as follows:

- Students define and create contiguous districts. All blocks in one district are connected to the next by at least one full side of a square.
- Students will persevere in problem solving. Creating districts that meet given criteria will require many drafts. Through this process, students learned the importance of approaching problems without having the final solution in mind.
- They will utilize proportionality as a way to measure fairness. In Squaretopia, 40% of the voters are in the yellow party, and 60% the blue party. Therefore when students created their proportional maps with ten districts, four were yellow, and six were blue.

- Students will identify the number of votes needed to win a district.
- Students will utilize packing and cracking practices to maximize the desired party's number of districts and minimize the other party's number of districts.
- They will discuss if a district is compact or not compact using the vocabulary of area and perimeter.

Students discovered many ways to create these ten districts. The blue party can win anywhere from 3 districts to all ten districts, depending on how these districts are created. There are 60 blue votes, and a minimum of 6 votes per district is needed to secure a majority. Theoretically, there are enough blue votes for the blue party to win all ten districts. Similarly, the 40 yellow votes are enough for yellow to win up to 6 districts. It is neither fair for the blue party to win all ten districts nor fair for the yellow party to win six districts. Students consider what they think is a fair amount of districts for each party to win. Should the number of districts won by each party be proportional to the total number of votes that each party received? Or, is fairness achieved by making compact districts? We explore these questions in the activities.

Student Work

We use a ten-by-ten grid of blue and yellow squares cut into ten districts of ten squares each. Districts with ten squares can result in a 5 to 5 tie. Thus, instructors may consider using 9x10 or 7x10 grids with an odd number of either 9 or 7 squares per district, which cannot result in a tie. The students had several options for creating their district maps. Ideas include using a Google drawing annotated with color labels or district numbers, drawing district boundaries using a tablet or written annotations or highlights on a color or grayscale handout. There are likely many other ways, and we look forward to exploring them in the future. The resulting maps consisted of ten individual shapes. Some students benefit from scaffolds to help visualize the individual shapes. For example, individual districts may be physically cut apart. Or, in a digital version of the activity, one can grey out all districts except one to draw focus.

In Squaretopia, students were able to explain and demonstrate packing and cracking. When discussing their grid of yellow and blue voters (Figure 4), a student wrote, "There is a way for the yellow to win. The trick is to minimize the blue; to use them up when making the groups. There are about two to three groups that are made out of 100% blue tiles. Then the other groups are made of mainly six yellows and four blues. This is how you can make a minority a majority." This student successfully cracked the blue party by finding the number of votes needed for yellow to win, which was 6; the number needed for blue to win; finding the remainder in the district, 4; and then spending four yellow votes in as many districts as possible, in this case, Districts 1, 3, 5, 7, 8 and 10. They also packed blue into Districts 4, 6, and 9.

Another student noted, "This is how you make yellow win: You pack four districts, all full of blue. The blue is winning the four districts by 100%. If you pack 40 blue into four districts, you have 20 blue left and 40 yellow. You use 40 yellow to spread out those remaining districts, and the remaining blue is just the minority over the yellow." Because there were 40 yellow voters and six needed to win a district, the maximum number of districts that yellow could win was six, since 40/6 = 6 with a remainder of 4. This, in turn, meant there would be a minimum of four blue districts. The student described packing blues into those four districts.

In general, student maps that are gerrymandered will yield very non-compact districts. Figure 5 (next page) is a student's map of a gerrymander for the yellow party.

Figure 4

Student Work Sample #1

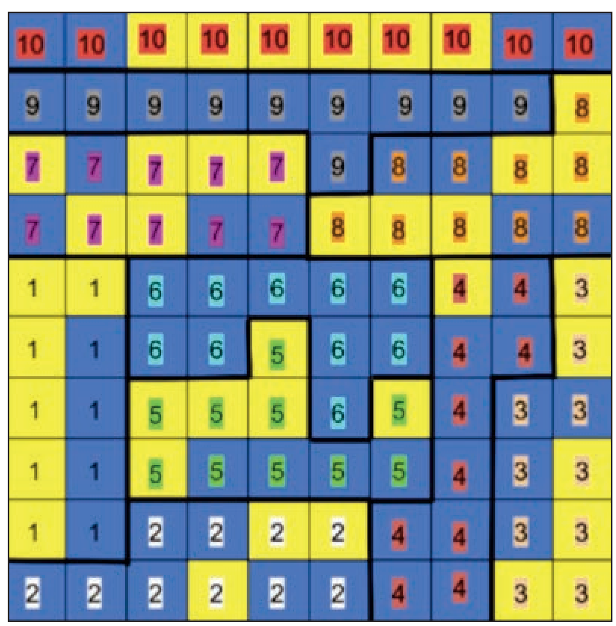
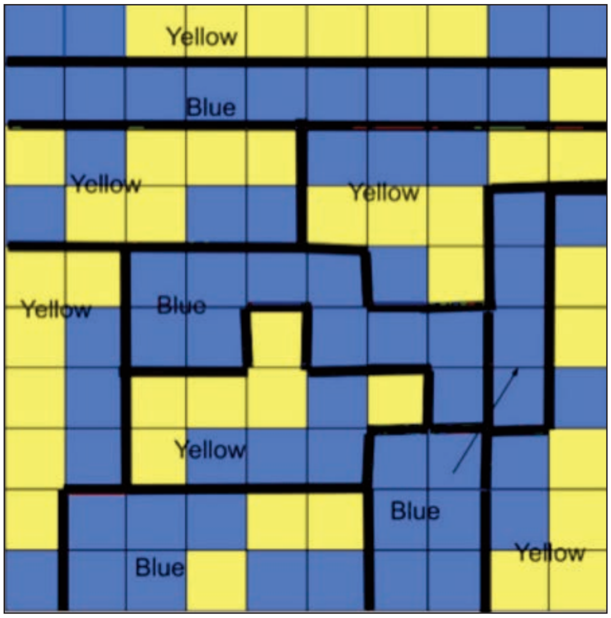


Figure 5

Student Work Sample #2



They note in an end-of-unit project that:

> Congress is causing this problem by gerrymandering each district state map so that the minority political party wins over the state and they get more seats in the House of Representatives. In the imaginary state called Squaretopia, blue is supposed to win because it makes up 60% of the state. My blue districts are really weirdly shaped because my objective was to make yellow win. Once I secured the win for the yellow, I just started grouping the remaining squares into districts. Congress does the same, and that is how gerrymandering happens.

This brings us to a key question of this unit: How can we best measure fairness? As we show in the next section, one possible metric is compactness.

Part Four: Calculating and Analyzing Compactness Metrics

In this section, the compactness metrics were presented to students, who then computed them for the districts they created in their Squaretopia maps. Our learning goals here are the following.

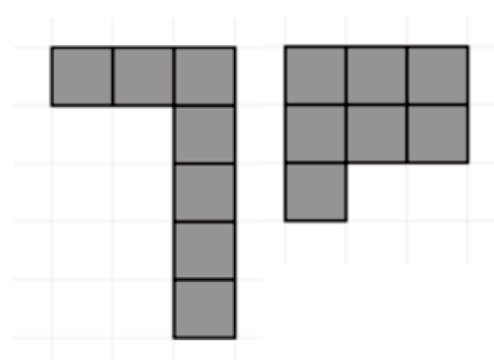
- Students will appropriately identify geometric features (area, perimeter, length, width) and evaluate compactness metrics in terms of these features.
- Students will discuss both abstractly and using numeric values what each metric is capturing. For example, the Skew Metric captures only a ratio of length to width, while Square Reock illustrates how close a shape is to being a square. (These metrics are explained at length in Appendix B.)
- Students will review each metric critically to analyze how accurately each metric captures compactness and discuss this analysis with peers and in writing.

Here are two cases where we see the Supreme Court directly reference compactness: "I know it when I see it" Supreme Court Karcher v. Daggett (1983) and "bizarrely shaped and far from compact" Supreme Court Bush v. Vera (1996).

In the previous section, we discussed the four Squaretopia maps which students created. The most unfair map was arguably the gerrymandered map for the yellow party since the party with the fewest overall votes should not win the majority of the districts. Students considered whether it was possible to tell by looking at the shape of the districts that the map is unfair. We introduced the concept of "compactness." Like the Supreme Court, students expressed that the gerrymandered maps have districts that look less compact. But what does that mean? Figure 6 shows an opening activity that we created for students to formulate their thoughts on what it means for a shape to be compact.

Figure 6

Classroom Activity



Argue that one of these districts is more "compact" than the other. Try to use specific geometric ideas to make your argument.

Argue that one of these districts is more "compact" than the other. Try to use specific geometric ideas to make your argument.

The following statements represent ways in which students intuitively defined compactness:

- They compare the length to the width.
- They identify the amount of "free space" and state that greater amounts of free space mean a shape is less compact.
- "The right one is two boxes away from being a square while the one on the left is eight boxes away from being a rectangle."
- "They both have seven boxes and three columns. There are less rows on the right, so it is more compact."

Compactness Metrics.

We now transition from students' intuitive arguments about compactness to the metrics that have been defined and used in gerrymandering legal cases. Over 24 compactness metrics have been used in the literature about gerrymandering. The metrics Skew, Isoperimetric, Square Reock, and Convex Hull are based on actual compactness metrics among the most widely used (Barnes & Solomon, 2020). These metrics were adapted for classroom use by MGGG (2017). In Appendix B, there are definitions of each metric. Each metric is designed to give higher values to shapes that are more compact and are in the form of a ratio comparing the area to the perimeter. The range of scores is (0, 1] with 1 being perfectly compact. The most compact shape is a circle, meaning that a circle has the most area for the least perimeter and minimizes the longest distance across the shape. The districts in Squaretopia have boundaries along the edges of a square grid, and thus the most compact district on the grid is a square. These compactness metrics have been altered to reward square shapes.

Students gather data for each of the four Squaretopia maps they created in Part Three: compact, proportional, gerrymander for blue, and gerrymander for yellow. The chart shown in Figure 7 provides a simple template of the information the students will gather.

The teacher may choose to provide four copies of this chart, have students choose their own organizational structure, or provide spreadsheets with built-in formulae to help ease some of the tedious number crunching. For a class on n students, this results in $40n$ values for each compactness metric which can be analyzed as a class. Having this information for each of the four maps can lead to a rich analysis of which features of the shape each metric is capturing.

Students then find evidence of gerrymandering. They had already computed one or all of the compactness metrics for a proposed map. Now, they look for deviations from the ideal result of 1.0. The map with metrics that are overall closer to 1.0 can be argued to be the least gerrymandered. We can see these justifications made by a student calculating and comparing Isoperimetric and Square Reock measures in Figure 8 (next page). They stated:

> Compactness metrics are mathematical methods used to distinguish if the shape is closer to being a square or not. A shape is considered compact when closer to a square. After getting the results for shape 1 (0.62) and shape 2 (0.81), we can say that shape two is more compact because shape 2 is closer to 1.0 than shape 1. Since shape 1 was an example taken from the gerrymandered map, we can easily say that if one district in that map isn't compact, then neither are the rest. Using the Square Reock method as well, we have another proof that shape two is more compact than shape one because it is still closer to 1.0.

Figure 7

Collecting Compactness Data

District	Skew	Isoperimetric	Square Reock	Convex Hull
1 2 – – – 10				
Average				

Figure 8

Student Work Sample #3

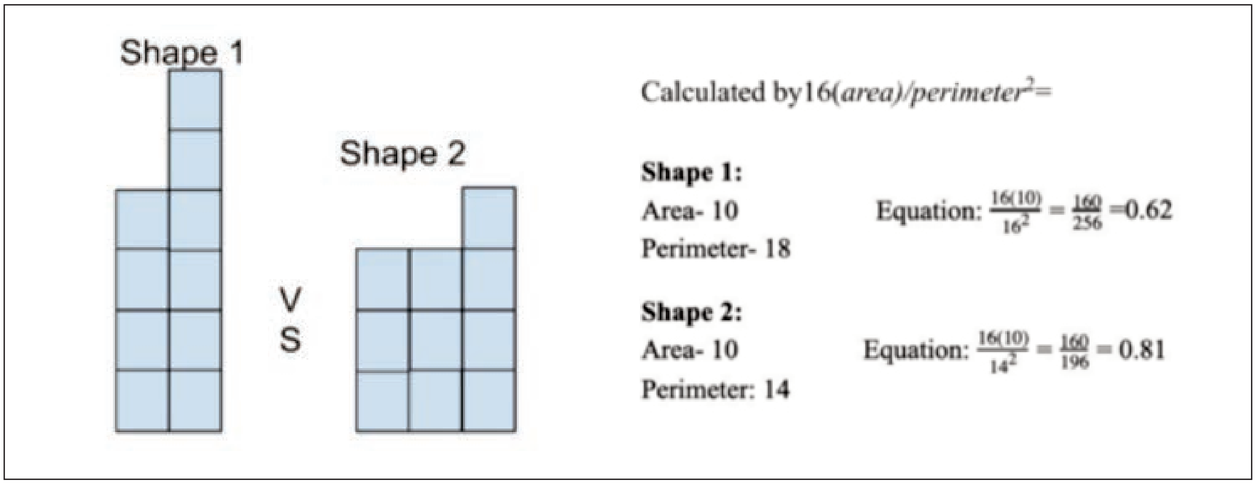
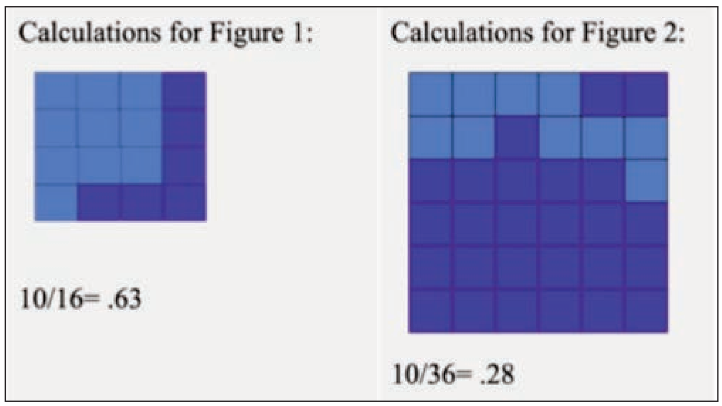


Figure 9 illustrates another student's analysis using Square Reock.

Figure 9

Student Work Sample #4



They shared that:

> The formula A/S (A = Area and S = Area of the smallest square containing district) Figure 1 is also proven to be more compact with the Square Reock method because Figure 1 is 63% compact and Figure 2 is 28% compact. This method really shows how gerrymandered Figure 2 is because it's only 28% compact.

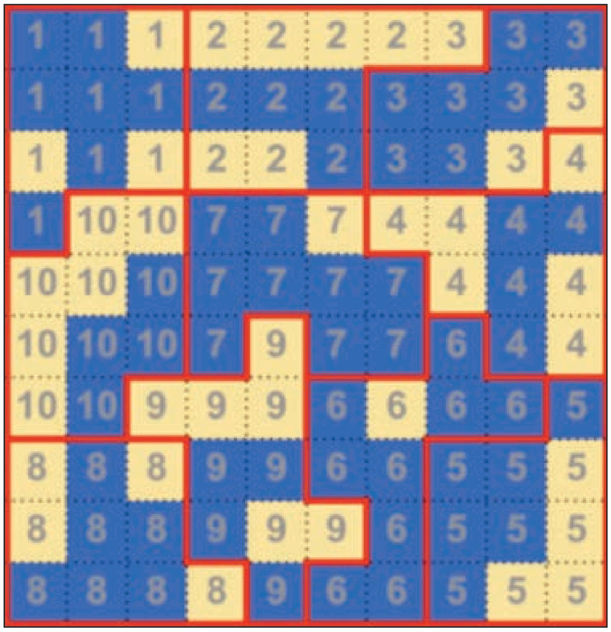
There is no communally agreed-upon way to prove gerrymandering (Barnes & Solomon, 2020). This realization came from the students' exploration and discussion with each other. We provided students with the opportunity to calculate the compactness metrics. Then, we facilitated a critical discussion about which measurement best proves gerrymandering, using the data from the chart in Figure 7. Students identified districts they had purposefully packed and cracked and then named the compactness metric that was furthest from 1.0 for those districts. Results were compared to come to a consensus about the best compactness metric. It is key to note here that consensus may not be achieved, and if it is, the decision about the best compactness metric is likely to be different for every class of students. Our true objective is the critical discussion taking place.

Using compactness metrics to argue about their Squaretopia maps, students were able to use prior knowledge about area and perimeter to explore a real-world topic. We saw in Figures 8 and 9 that gerrymandered districts are likely to be non-compact. However, the inverse is not true. Non-compactness does not imply gerrymandering, despite the use of this argument in the public gerrymandering conversation. We can see in Figure 10 (next page) that a proportional map is not necessarily compact.

This map may appear to have evidence of *hijacking* and *kidnapping* (in which an incumbent's address is carved out of their district or two incumbents are pushed into the same district). In Districts 6 and 9, we see concavity and lasso-like boundaries that could support this theory. But in reality, the boundaries of this map were drawn to result in proportional representation. This leaves us with the question: If compactness metrics cannot definitively prove gerrymandering, what other tools can we use? This is a question still to be answered.

Figure 10

Student Work Sample #5



Conclusion and Student Reflections

Our goals for this unit are that students (a) discuss how to quantify fairness, (b) analyze a formula and evaluate whether it captures what it claims to capture, and (c) discuss compactness and features of districts both generally, by using mathematical vocabulary, and specifically, by computing numeric calculations. Through their written arguments, students quantified fairness. Students made the connection between disproportionality and unfairness, pointing out that a party with 40% of the vote should not get 60% of the representation. For further exploration, we suggest a lesson about the Efficiency Gap, which compares the number of wasted votes for each party (See Bernstein & Duchin, 2017). We saw students identify their choice for the strongest compactness metric in the materials presented here by seeing which metric deviated most from 1.0 for the gerrymandered districts they had created. In future classroom iterations, we hope to invite students to develop their own methods to measure compactness that can be applied to their Squaretopia maps. We are curious to find out if the methods developed by students might be quite similar to some of the accepted compactness metrics. If students are engaged in creating the metric, they will also be thinking about how to best capture the minimization of the perimeter for a given area. Having authored a formula themselves, they will be better equipped to critique existing formulas. In the short classroom activity in Figure 6, students used area and perimeter generally and numerically by pointing out how some shapes used lots of extra perimeter for the same area. We believe this activity could be expanded to allow students to define such a metric themselves.

We hope that when new district maps based on 2020 Census data are presented in the media, our students will be able to engage in an informed discussion about them. Because of our unit, students can observe that gerrymandering leads to a disproportionate partisan split. They can look at maps, identify compactness and discuss implications. As math teachers, we are accustomed to our students having math anxiety or assuming that class content will not be used outside of the classroom. A student reflected, "Usually when I learn math, the questions that go through my head constantly are 'When will I ever use this in real life?' and 'How does this relate to anything in the real world?'" Math-phobia extends all the way to the Supreme Court. There have been several cases where the highest court has pushed back against mathematics presented in cases. In response to mathematical presentations on gerrymandering, Chief Justice Roberts said, "It may be simply my educational background, but I can only describe it as sociological gobbledygook." Justice Breyer asked, "Is there a way of reducing it to something that's manageable?" (Roeder, 2017, para. 8, 14). Our purpose here is to unpack the 'gobbledygook' and to provide students with opportunities to reason and think critically about problems that affect all of us. As a student wrote, "We got to bring what's going on in the real world into the classroom. It was pretty interesting that we could use math to solve problems that are going on around the country." Our students engaged in dialogue about the workings of our government and utilized quantitative reasoning to identify unfair representation. The following quotations were chosen to highlight students' reflections about engaging with this unit:

> "You can have a state that is 35% blue, yet gerrymandering can find a way to make blue the winning party... it rigs elections and destroys the idea of democracy, by citizens not getting what they asked for."

> "I learned a lot about how it affects me and the offices that speak up for me… I now truly understand how district maps are made and how the House of Representatives works. I didn't know that the number of seats each state is given depends on the population."

"I think it will help me a lot in my life because this is a real-life problem that affects all of us, meaning the minorities. It was great to learn about this, to be aware of what is happening out there, and to join other people that are trying to stop this from happening because it is not fair."

"You have no idea how many times I've explained gerrymandering to people since I got to college [...] you'd be genuinely surprised how many people are completely unaware of gerrymandering and how greatly it affects our political climate."

Teaching gerrymandering allows students to reason through mathematical concepts of area, perimeter, and proportionality within a meaningful and relevant context of equity and representation. We have not yet solved the problem of gerrymandering. We need more mathematical thinkers to be a part of the conversation and contribute to creating fair representation in our government. By creating, analyzing, and critiquing Squaretopia maps, students learn that math is not about getting the right answer but using mathematical tools to communicate and make arguments. This process allows students to better understand what mathematics is and creates opportunities for using math as a tool in a myriad of ways.

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Appendix A

Guiding Questions for Part One: Contextualization

1. What is a district? What types of districts exist in the United States for federal, state or local government?
2. In terms of population, which are big states, and which are small?
3. How should each state be represented in the federal government? Should we give the same number of representatives to each state or should the number depend on the population of the state?
4. In the House, where the number of representatives is determined by population, how many should each state get?
5. How many total seats are there in the House? Is this number fixed? Has it always been?
6. What is the partisan split of the state?
7. How should the number of Democratic and Republication representatives relate to the partisan split of the state?
8. What is the population density of each district and state and how is this shown in maps?

Appendix B

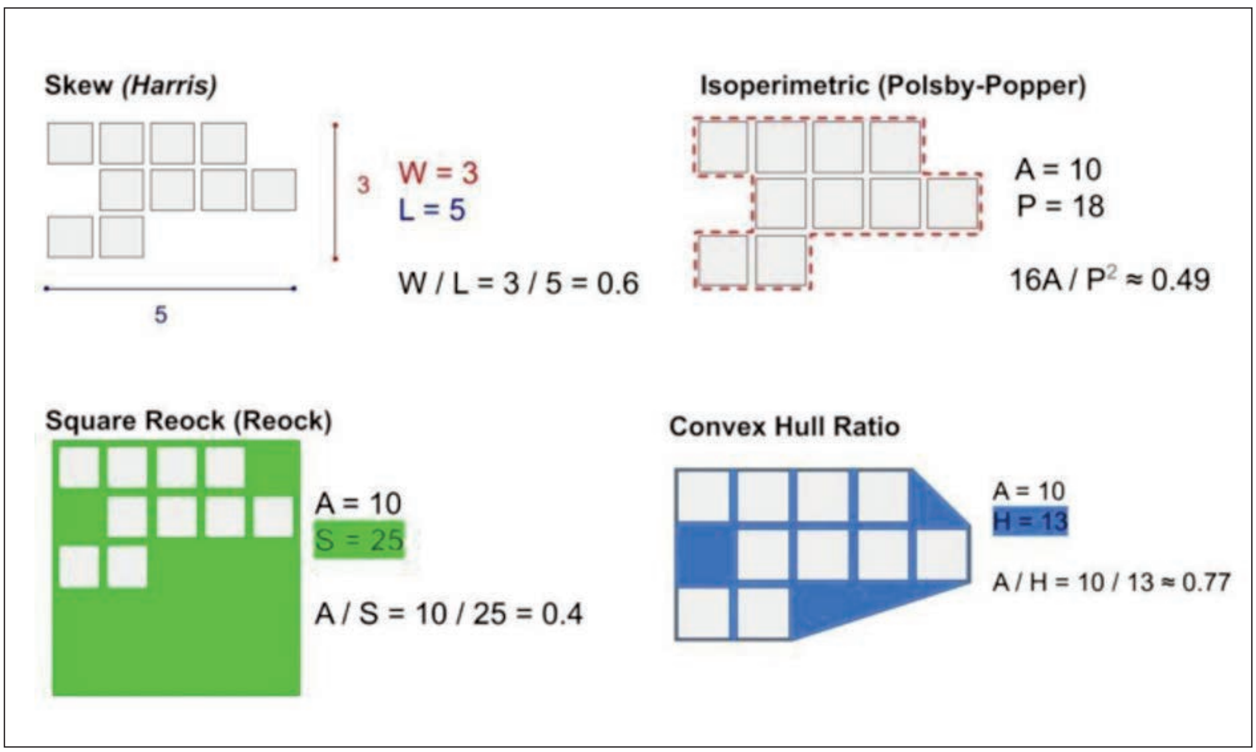
Compactness Metrics

The diagrams in Figure 11 were created by Bernstein, a founding member of the Metric Geometry and Gerrymandering Group. These metrics were introduced to us at the MGGG conference (2017).

- **Skew**
  W / L, where W = shorter dimension, L = longer dimension
  - Adapted from Harris, in which L = longest axis, W = greatest width perpendicular to that axis
  - It can be argued that this measurement rewards maps where districts are as compact as possible. However, there are some tricks, like X or L shapes.
- **Isoperimetric**
  $16A / P^2$, where A = area, P = perimeter
  - Adapted from Polsby-Popper: $4\pi A / P^2$
  - Consider a circle. We know the circle is the shape which provides the maximum area for a given perimeter. Duchin points out, "plumper things have more area" (MGGG, 2017). The Polsby-Popper metric establishes the circle as the most efficient shape.

Figure 11

Compactness Metrics, Bernstein, 2017



- $4\pi A / P^2 = 4(r^2) / (2r)^2 = 1$
- In Squaretopia, we are constricted by right angles, and thus our Isoperimetric calculation is adapted to demonstrate a square as the most efficient shape.
- $16A / P^2 = 16 \times side^2 / (4 \times side)^2 = 1$

- **Square Reock**
  A / S, where A = area, S = area of smallest square containing district
  - Adapted from Reock: A / C, where A = area, C = area of smallest circle containing district
  - Like Isoperimetric and Polsby-Popper, this measurement rewards maps where districts are as close to the most efficient shape as possible.

- **Convex Hull**
  A / H, where A = area, H = area of convex hull
  - When imagining the convex hull for a shape, it may be helpful to use physical or virtual geoboards (a plastic grid with pegs placed at each intersection of vertical and horizontal lines) and have students snap rubber bands around the district.
  - This measurement rewards maps where districts have minimized divots and pivots that usually accompany the practices of kidnapping and hijacking.

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Hyper-acceleration of Algebra I: Diminishing Opportunities to Learn in Secondary Mathematics

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ABSTRACT An increasing number of students are *hyper-accelerating* their study of formal Algebra I to Grade 7 or earlier to maintain a competitive edge in the race to calculus. However, there is a lack of evidence that these students construct the conceptual foundations necessary for persistence in advanced mathematics. This paper maps the drive toward hyper-acceleration to the historical underpinnings of acceleration of Algebra I to Grade 8. We illuminate how acceleration can detract from opportunities for middle school students to engage in algebraic reasoning in preparation for advanced mathematics. We further describe how this pathway exacerbates persistent inequities in secondary mathematics education. Our synthesis of the literature on Algebra I acceleration, readiness for STEM undergraduate study, and equitable access is the basis for our argument for more research on hyper-acceleration.

KEYWORDS *algebra, calculus, hyper-acceleration, opportunity to learn, equity*

Students who take Algebra I in Grade 8 have more opportunities to take calculus in high school. The presumption is that they will have more access to rigorous mathematics experiences and more success in science, technology, engineering, and mathematics (STEM) in college. At the same time, *hyper-acceleration* of Algebra I to Grade 7 or earlier has emerged as a form of curricular intensification in secondary mathematics (Domina & Saldana, 2012). This further acceleration is in response to parent and administrator pressures for students to take calculus in their junior year of high school to improve competitiveness for college admissions (Bressoud, 2017). Hyper-acceleration is consistent with Lucas' (2001) theory of effectively maintained inequality in which "advantaged actors secure for themselves and their children some degree of advantage wherever advantages are commonly possible" (p. 1652). As an increasing number of students have access to Grade 8 Algebra I, privileged stakeholders will pursue further Algebra I acceleration as an educational advantage.

While school communities may idealize hyper-acceleration as an indicator of smartness and status, its implications for mathematics learning are frequently questioned in education blogs and editorials. For example, Kaplinsky (2017) and Pemantle (2016) challenged acceleration policies that compact middle school standards and diminish opportunities to learn foundational content for advanced mathematics. Picciotto (2014) argued that hyper-acceleration is motivated by the belief that "kids from certain families are just better at math and deserve the various advantages supposedly conferred by being 'ahead'" (para. 3). This pathway is inconsistent, however, with expert recommendations about the judicious acceleration of algebra and deep mathematics learning. The National Council of Teachers of Mathematics (2016) cautioned that students who are talented or express strong interests in mathematics should not rush through critical concepts. Sheffield (2017) argued similarly that the appropriateness of acceleration of secondary mathematics by more than one year for gifted students is a "dangerous myth" (p. 21) and "not beneficial for a majority of top students" (p. 22).

Despite these admonitions, the pervasive drive to complete more Advanced Placement mathematics courses makes it unlikely that students, parents, and teachers will support a reversal of this trend toward taking Algebra I at younger ages (Bressoud, 2020; NCTM, 2018). In addition to enrolling in higher course levels, students on hyper-accelerated Algebra I pathways are presumed to have access to more rigorous content and more qualified teachers. According to Tate (2004), quality of instructional delivery, content emphasis, content exposure, and coverage are measures of opportunity to learn (OTL) in mathematics. Using these metrics, hyper-accelerated students may appear to have a greater OTL because they enroll in and complete more secondary mathematics courses. However, lost opportunities for meaningful mathematical reasoning can be hidden behind the presumed advantages of hyper-acceleration. Hyper-accelerated students may experience mathematics as a competitive hierarchy instead of as a creative sense-making endeavor (Galanti, 2021).

Because the empirical research on hyper-acceleration is limited, we synthesize literature on Grade 8 Algebra I and readiness for STEM undergraduate study to contextualize this phenomenon. We also advocate for additional research on OTL with hyper-acceleration at both individual and systemic levels. In what follows, we first present a brief history of Algebra I acceleration. Then, we discuss the evidence that hyper-acceleration may not adequately prepare middle school students for advanced mathematics. Next, we argue that acceleration can detract from opportunities for students to engage in algebraic reasoning as a sense-making endeavor in the early grades. Finally, we describe how this pathway exacerbates persistent inequities in access to high-quality secondary mathematics education.

History of Algebra I Acceleration

Algebra I in Grade 8 or Earlier

Over the last 30 years, access to Algebra I in Grade 8 has increased with the goal of improved student achievement and college readiness (Clotfelter et al., 2015). The enrollment of U.S. students in Algebra I in middle school grew from 16% in 1990 to 47% in 2011 (Loveless, 2013). We argue that this increase explains the sociological motivations for hyper-acceleration. These motivations are also at the root of issues of equity and access in secondary mathematics education. Before the early 1990's, access to Algebra I in Grade 8 was reserved for a small percentage of students who demonstrated strong aptitude in pre-algebra. Moses (1995), however, challenged mathematics education researchers at the Algebra Initiative Colloquium to view algebra as the new civil right for students from all backgrounds. Participants debated the assumed challenge of reforming curriculum and pedagogy with this increased access. They wondered whether "algebra for all" would lessen the rigor of Algebra I content for an increasingly diverse community of learners. Lacampagne (1995) remarks:

> A question that plagued Colloquium participants was, "How do we ensure that 'algebra for all' is not 'dumbing down' algebra?" The mathematical community as well as parents of college-bound students will and should demand sound preparation in algebra for the college bound. We will be faced with building an algebra curriculum and pedagogy that will support the needs of all students. (p. 4)

This statement suggests that broader access to Algebra I could threaten the privileged role of formal mathematics education in identifying the elite students of the future. These concerns foreshadowed the emergence of hyper-acceleration of Algebra I to earlier grades.

Empirical studies broadly define Algebra I acceleration for Grade 8 or earlier. There is no direct focus on hyper-acceleration; however, inferences about the expansion of Algebra I in Grade 7 can be made from a series of relevant studies. Many school divisions offer Algebra I, Geometry, and Algebra II in a linear sequence between Grades 7 and 9. Domina et al. (2016) found that between 2003 and 2013, the percentage of students enrolled in Grade 8 Geometry in California more than tripled from 2% to 7% in parallel with "algebra for all" initiatives. This indicates that the number of students taking Algebra I in Grade 7 increased as more students gained access to Algebra I in Grade 8. Moreover, our analysis of course-taking data from the Characteristics of Successful Programs in College Calculus (CSPCC) study (Mathematical Association of America [MAA], 2017) shows that 12.8% of freshman-level calculus students had completed both Geometry and Algebra II by the end of Grade 9. Based on this finding, we inferred that these college freshmen took Algebra I in Grade 7.

Algebra I Acceleration Outcomes

The acceleration of Algebra I to Grade 8 has a history of mixed outcomes for students. For instance, accelerated Algebra I has been associated with higher standardized test performance, grades, and college enrollment (Gamoran & Hannigan, 2000; Smith, 1996; Spielhagen, 2006; Stein et al., 2011). However, other factors such as gifted status, school context, school attendance, and parents' education levels contributed substantially to differential outcomes in achievement between non-accelerated and accelerated students (Rickles, 2013). Furthermore, universal Grade 8 Algebra I policies in states like California and North Carolina had adverse effects (Clotfelter et al., 2015; Domina et al., 2015; Finkelstein et al., 2013; Remillard et al., 2017). For example, Clotfelter and colleagues (2015) reported that acceleration has "statistically significant harmful effects" (p. 180) based on end-of-course test scores in Algebra I, Geometry, and Algebra II. Adverse effects were reported for up to the 60th percentile of the Grade 6 and 7 mathematics achievement distribution. Broad Algebra I acceleration has also resulted in students repeating mathematics courses (Finkelstein et al., 2014; Lee & Mao, 2020) and students exiting the mathematics pipeline before their senior year of high school (Finkelstein et al., 2014). Our descriptive analysis of transcript data presented in the Finkelstein et al. (2014) study revealed that nearly half of the students who studied Algebra I in Grade 7 repeated the course in Grade 8.

These findings challenge the assumptions that Algebra I acceleration equalizes outcomes and increases access to advanced mathematics courses. They also suggest that such outcomes could be worsened by further accelerating Algebra I for students who have not yet built strong pre-algebraic foundations. Hyper-acceleration may not be fostering the mathematical understanding and confidence that many students need to be successful on advanced secondary mathematics pathways.

Early Algebra vs. Algebra Early

The shift of Algebra I to the middle grades is part of a continuing conversation in mathematics education about early algebra. Contextual sense-making grounded in algebraic reasoning and generalization from arithmetic is often referred to as early algebra (Carraher et al., 2006; Stephens et al., 2017). Kaput (2008) argued that early algebra could "democratize access to powerful ideas by transforming algebra from an inadvertent engine of inequity to a deliberate engine of mathematical power" (p. 6). Yet experts in this field of study distinguish early algebra from Algebra I studied in the early grades. Early algebra emphasizes background contexts in problems while gradually introducing formal notation (Carraher et al., 2008; Mason, 2017). These experiences are fundamental to understanding abstraction and structure within Algebra I as variables and symbolic notation are formally introduced (Driscoll, 1999). The construction of deeper understandings of rational numbers and proportional reasoning in the middle grades before formal algebra courses at the high school level is also crucial (Sheffield, 2017). Taking Algebra I in Grade 8 or earlier may adversely impact students' productive experiences with early algebra and their conceptions of ratio and proportion. Because of societal beliefs that early Algebra I provides a competitive advantage in college admissions, many students may accelerate toward advanced mathematics at the expense of meaningful algebraic reasoning in the elementary and middle grades.

Accelerated Algebra I and STEM Readiness

A common rationale for Algebra I acceleration is the perception of improved readiness for undergraduate STEM study. Taking Algebra I in Grade 8 has also been associated with taking calculus in high school and increasing the likelihood of STEM undergraduate study (Lee & Mao, 2020; Rickles, 2013). Many students believe that they will be at a disadvantage in college if they have not studied calculus in high school (Bressoud, 2020). Despite the increase in the number of students who are accelerating their study of Algebra I as a pathway to high school calculus, research has shown that many high school graduates are not prepared to succeed in undergraduate mathematics. Only 20% of the 2019 high school graduating class met the American College Testing (ACT) STEM readiness benchmark (ACT, 2019). This benchmark is derived from both mathematics and science subscores and college student performance data. MAA and NCTM warned that the increasing acceleration of traditional secondary mathematics courses is not only ineffective but counterproductive in building foundational mathematical knowledge for a STEM career (Bressoud et al., 2012). According to Stewart and Reeder (2017), many students struggle in college-level mathematics because of incomplete or insecure understandings of algebraic topics situated within middle school and high school curricula.

Multiple studies quantify the impact of Algebra I acceleration when students do not build the requisite confidence and conceptual understandings necessary for success in college STEM courses. Sadler and Sonnert (2018) reported that end-of-course grades in high school precalculus courses along with SAT/ACT scores explained more than twice the variability in college calculus performance than grades in high school calculus courses. The National Council for Education Statistics (2013) reported, meanwhile, that 13.5% of students who completed high school calculus enrolled in remedial mathematics in college. Additionally, the Factors Influencing Calculus Success in Mathematics (FICSMath) study showed that 20% of calculus students who first enrolled in a college precalculus course had already completed high school calculus (Sonnert & Sadler, 2014). In the CSPCC study of over 14,000 students who enrolled in an entry-level calculus course required for STEM majors, 67% had studied calculus in high school. 36% of students who earned a three or higher on their AP Calculus exams earned a "C" or lower in college calculus. The grade distribution for students who had earned less than three on their AP Calculus exams was comparable to that of students who did not take calculus in high school (Bressoud, 2015). The FICSMath and CSPCC studies did not segregate the data on Algebra I course-taking by grade level; however, the findings should raise questions about the consequences of hyper-acceleration and the loss of not one but two years to build foundational mathematical understandings.

Algebra I Acceleration as a Matter of Equity

The negative consequences of Algebra I in Grade 8 or earlier for individual students must also be accompanied by a broader critique of acceleration as a form of tracking that exacerbates systemic inequities in middle school mathematics. Schmidt (2009) examined OTL as content coverage in Grade 8 mathematics courses using teacher survey data on the Trends in International Mathematics and Science Study (TIMSS). He attributed 40% of the variation in mathematics achievement to differences in content coverage as a result of tracking. Stein and colleagues (2011) analyzed both universal and selective Algebra I acceleration policies. They suggested that students who would have been excluded from algebra under the "old rules" might be grouped and taught a less rigorous version of algebra. Open-enrollment policies led many middle schools to create multiple algebra courses. More recently, analysis of OTL using student reports of mathematical content coverage within the 2012 Programme for International Student Assessment (PISA) data showed that the greatest differences in OTL in the United States occurred *within* schools and not *between* schools (Schmidt et al., 2015). These persistent educational inequities affirm Schmidt's (2009) earlier study of OTL in Grade 8 mathematics in which he concluded that only the highest-achieving students benefit in tracked schools. As we look across these findings, we can infer that hyper-acceleration intensifies the stratification of middle school mathematics courses and lowers content expectations for students who struggle.

Hyper-acceleration has moved the historical gatekeeping role of Algebra I to an even younger age with no empirical justification. This has contributed to further stratification along race and class lines, as narrow constructions of mathematics ability and achievement are often defined by race and class (Boaler, 1997; Boaler & Greeno, 2000; Gutiérrez, 2012; Louie, 2017). These biases continue to be reflected in the overrepresentation of White and Asian students in accelerated Algebra I classrooms (Education Trust, 2020; Grissom & Redding, 2016). Stinson (2004) asked, "How might mathematics educators ensure that gatekeeping mathematics becomes an inclusive instrument for empowerment rather than an exclusive instrument for stratification?" (p. 8). We must interrogate hyper-acceleration as a structure that undermines student learning and exacerbates the underrepresentation of other racial and ethnic groups in accelerated secondary mathematics (Irizarry, 2020; McCallum & Novak, 2020; Morton & Riegle-Crumb, 2020). Hyper-acceleration diminishes meaningful learning when mathematical success is defined by faster course completion. Secondary mathematics experiences should instead foster creative problem solving and persistence in mathematics for all students.

Directions for Future Research

Research on the acceleration of Algebra I is often framed in terms of increased access to rigorous mathematics in high school. However, Algebra I acceleration as policy cannot be separated from the equally important obligation to build a foundation for continued course taking and productive dispositions in mathematics. There are empirical contradictions between studies that relate improved secondary outcomes with the increased acceleration of Algebra I and studies that challenge acceleration as detrimental to building strong precalculus understandings. The mathematics education community needs to unpack these contradictions as we strive for more heterogeneous learning environments and position more students to succeed in advanced mathematics courses. Ideally, hyper-acceleration should motivate and empower students with mathematical talent, creativity, and passion irrespective of race, class, or economic status. Instead, it introduces the potential to devalue early algebra experiences and reify existing societal power structures.

Our synthesis of research on Grade 8 Algebra allows us to make inferences about the risks of hyper-acceleration. Still, there is a need for further research to understand this phenomenon. It is concerning that the presumed educational advantage of offering hyper-accelerated Algebra I within a school might become more important than increasing opportunities for meaningful algebraic reasoning for every student. Large-scale research using test scores, course completion, and advancement toward college can hide the individual and systemic implications of hyper-acceleration. This research must be accompanied by an examination of the quality of learning within these contexts. The increasing stratification of secondary mathematics courses demands critical questions about who has access to this social capital and how student identities and backgrounds predict participation in these courses.

The following research questions can quantify structural disparities and student outcomes from a content perspective on OTL:

- How are students identified and selected for hyper-accelerated Algebra I?
- How does hyper-acceleration relate to further mathematics course taking and undergraduate STEM participation?

There is also a need to look beyond traditional OTL metrics of content coverage and delivery of instruction. A situative perspective on OTL (Greeno & Gresalfi, 2008) captures the complexities of interactions amongst students, parents, teachers, administrators, and curriculum. OTL can thus be described as a longitudinal trajectory of mathematics participation with a past, a present, and a future. This situative perspective can illuminate how differential social status associated with hyper-acceleration relates to classroom participation, identities of competence, and persistence in mathematics. The following additional research questions can elicit the contextualized and affective aspects of hyper-acceleration from a situative perspective on OTL:

- How do community stakeholders (e.g., students, parents, teachers, and administrators) describe the appropriate acceleration of Algebra I?
- What opportunities do students have to make sense of algebraic content in hyper-accelerated Algebra I classrooms?
- How does hyper-acceleration relate to a student's evolving sense of mathematical competence?

Conclusion

In this paper, we presented literature and data about the acceleration of Algebra I and the related adverse outcomes. The contradictions raised in this article should motivate the empirical investigation of hyper-acceleration as both policy and practice. Building on these contradictions, we offered potential paths for future research. Suppose we continue to encourage hyper-acceleration as a social marker of distinction, without evidence of its individual and systemic impacts. In that case, we will lose ground on building a more diverse community of mathematically promising students (NCTM, 2016). It is time for the field to engage in a scholarly examination of hyper-acceleration related to conceptual understanding, persistence in advanced mathematics, and more equitable ideas about what it means to be "good" at mathematics.

Questions about hyper-acceleration are timely, and they relate to opportunities for all students to engage in rigorous secondary mathematics. These questions will become more critical as we emerge from the coronavirus pandemic and its legacy of differential access to high-quality mathematics teaching and learning. The June 2020 joint position statement from NCTM and the National Council of Supervisors of Mathematics (NCSM), *Moving Forward: Mathematics Learning in the Era of COVID-19*, reiterated earlier calls for detracking in the form of heterogeneous groupings in middle school mathematics classrooms (NCTM, 2018). These detracking efforts have become even more crucial in the COVID-19 era as school closures, absenteeism, and unequal access to technology exacerbate long-standing inequities and biases in assessing mathematical readiness (NCTM & NCSM, 2020; TODOS: Mathematics for All, 2020). Efforts to challenge and dismantle tracking (Berry, 2018; NCSM, 2019) cannot be successful without a critical examination of hyper-acceleration.

The perception that faster is better will continue to drive the political discourse in high-achieving school districts in the absence of new knowledge about the unintended consequences of hyper-acceleration. By arguably narrow processes for identifying students, hyper-acceleration can create structural barriers to learning not only for those who struggle within these new tracks but also for those who operate outside of these tracks. School stakeholders need empirical evidence to make informed decisions when faced with community pressures for this further acceleration of Algebra I. If offering increasingly advanced mathematics courses in middle school remains grounded in the desire of students, parents, and teachers to gain a competitive advantage in college admissions, we will see the continued growth in this phenomenon and, in turn, the growth of structural inequality in mathematics education. As we build an empirical basis for hyper-acceleration, we will move our field forward in new ways that inspire more meaningful participation in advanced mathematics for all students.

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NOTES FROM THE FIELD

Humanity and Practicality During the Emergency Conversion to Online Learning

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When the pandemic started in Spring 2020, I was one month into teaching two classes: a standards-based grading version of integral calculus and a project-based course on mathematical modeling. My approach for the emergency conversion to online learning involved choosing humane and practical options at every step to get everyone through the classes despite the difficulties. The first thing I did was reduce expectations for my students and myself in both courses to allow time for self care. In addition, I gave students the flexibility that they needed to weather the eventual sickness and deaths in our communities.

In the modeling course, we moved directly to the material on epidemiology. What a motivation for learning course material during a pandemic! Students collaborated in groups to develop computer simulation models of the spread of disease in the real world. Class consisted of my checking in with each group for about five to seven minutes to help students make progress on their projects. I pared the list of standards to the essentials in the calculus course by eliminating less-applicable concepts and simplifying the scoring system to a "pass" versus "progressing" dichotomy. This enabled students to focus on learning the material instead of worrying about their grades.

In transitioning to remote learning, we needed practical online replacements for face-to-face interaction. The online discussion board Campuswire allowed students to interact with each other outside of class, including a forum where they could ask and answer each other's questions. I received more direct messages on Campuswire than email messages, which made me believe that Campuswire reduced the friction of communication among us.

In addition to building community through Campuswire, I used Flipgrid for students to present and exchange feedback on their projects. I plan to continue using this video discussion tool. Before the pandemic, we would have to rush to fit all the presentations into one class period and deal with the technology issues that always show up during transitions. My students now use Flipgrid to record and re-record their presentations as desired, which are therefore much better prepared. Fellow students give dedicated feedback on these presentations, which we never had time for before.

I also leveraged Flipgrid for my Fall 2020 course in mathematical computing. In one project, students used Mathematica to design, prototype, and print a 3D sculpture using three-dimensional coordinate systems, mathematical transformations, and functional programming. When students received their 3D printed model in the mail, I asked them to record themselves unboxing their models and share them on Flipgrid. The students opened their packages and shared their excitement about seeing their project for the first time. Sharing this moment of humanity was a highlight for everyone.

Some students shared that they valued the empathy-based structure of my classes, including one who remarked, "I really enjoyed getting up every Monday and Wednesday morning to be part of your class. It really did brighten up my day in these hard times. Thank you for being a caring professor."

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NOTES FROM THE FIELD

COVID and the Importance of Casual Interactions in Mathematics Classrooms

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This year presented unimaginable obstacles to learning, some obvious and others more subtle. Even when students have good Wi-Fi, working devices, and the opportunity to learn in person—which most of my students have—there are barriers to learning that leave students feeling isolated, overwhelmed, and unmotivated. The most significant obstacle in my teaching, the physical distancing in the classroom, taught me how students learn. It has clarified the centrality of social interactions to mathematical learning.

In the times before COVID, I walked around and peeked over students' shoulders as they worked together on problems. I nudged them along at times by asking questions or providing hints. In graduate school, I learned that these casual interactions among students and teachers were at the heart of students' learning experiences. Through these interactions, students construct mathematical knowledge. Researchers have known for a long time that mathematical learning is active and social. Now we have stark evidence of the profound truth of this observation. These casual interactions in the classroom happen when students notice each other's work, overhear each other's conversations and talk to their neighbors. The interactions expose students to new ideas and new mathematical representations, and they spark creativity. Students also often disagree or come to different conclusions, which challenges them to justify their views. Casual interactions also allow students to try out ideas without committing to them fully. Students may take an intellectual risk by asking a question, showing scratchwork to a teacher, or offering a tentative idea to a neighbor.

This year, by contrast, students are tiny islands in the classroom. They are evenly distanced from each other and their teacher, sometimes with a plexiglass screen between them. There are no casual interactions. Every communication requires a deliberate, visibly raised hand and a commitment to enunciate clearly through a mask. The distancing creates a buffer that slows down the normal exchange of ideas and discourages risk-taking. Because of the effort required to speak to the other members of the classroom, only the most confident students volunteer, and they express only fully-formed ideas. It is not apparent, but the distancing discourages risk-taking even on paper. A student might use scratch paper to work out a problem in normal times, knowing that they would recycle the paper at the end of class. If the teacher saw incorrect work on a student's scratch paper, there was no problem. It was just an idea, just a draft. Now, there are no casual glances at a piece of work. Now, with every document communicated digitally, every piece of work feels more important, more permanent. Students hesitate to take a guess.

As difficult as the year has been, I am grateful for the wisdom and clarity it has given me and look forward to the opportunity to huddle and do math problems with my students once again.

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NOTES FROM THE FIELD

Meeting the Social-Emotional Needs of My Students During the Pandemic Through the Use of Activity Lists

Michelle Longhitano
Teachers College, Columbia University

Teaching mathematics during the pandemic as a high school teacher and instructional coach has been a transformative experience in prioritizing my students' social-emotional needs. To teach effectively in a virtual environment, I employed my knowledge of new and familiar technology—both to connect with students and to help them connect to each other. This was important since students expressed feelings of isolation from the lack of daily in-person contact in school. Social-emotional learning (SEL) has now become a priority in my approach to teaching mathematics. I felt this was essential for students' sense of belonging and emotional readiness to learn mathematics.

I attended workshops on hybrid teaching in the summer of 2020. My greatest takeaway from the training was the use of "activity lists." An activity list is a set of differentiated tasks that students can choose from to engage in the content at their own pace. In the 2020-2021 school year, I adapted and implemented activity lists in my classroom and used them as a vehicle for SEL. My activity lists are Google Docs that provide students with instructions and links to resources for each lesson. The lists also include learning objectives, check-in or kick-off activities, and written reflections about the content. Students can choose a modality to learn content (i.e., instructional videos, readings, small-group mini lessons in a separate Google Meet) and optional activities to extend the learning. I include SEL activities (e.g., guided meditations, mood meters) in the check-in portion of the activity lists before introducing new content. These activities built a community in my classes and provided a forum for students to address their emotional health.

In March of 2021, I surveyed the students to get general feedback about the course format. They expressed that they found comfort in consistency and options available for engaging in the course. A fully remote student commented, "I really like this format and having the ability to choose how I learn. It makes it easier also having a separate link available [to attend small-group mini lessons] for any questions that come up. I love the check-ins that we do." A hybrid student commented, "I feel like, this year especially, structure is needed, and I think this class in specific does have structure."

In summary, my use of activity lists has created an instructional experience for students that has enabled them to engage in SEL activities and control their learning by choosing a modality, path, and pace that meets their needs and learning preferences. In addition, this approach provided the structure and consistency students expressed was needed during this difficult time. Post-pandemic, I plan to continue my use of activity lists with an emphasis on SEL.

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NOTES FROM THE FIELD

A Digital Touch to Teaching and Learning Mathematics

Bryan Nevarez
Queens College,
City University of New York

Being thrust into remote teaching at the onset of the pandemic proved a challenging transition. I was fortunate to have a large whiteboard at home, a few dry-erase markers, and a 2014 MacBook Pro. I immediately rearranged the furniture in my basement to create my new classroom. After only two weeks of using Zoom and Google Meet for my classes, I realized that my makeshift setup was not working well. There were too many instances when students told me to move my camera "a little to the left (or to the right)" because light caused a glare on the reflective surface of the whiteboard. Poor Wi-Fi connection produced intermittent video streaming and quickly became a nuisance. Still, I knew there was a way to carry out the teaching and learning of mathematics online, despite the grim circumstances that we faced.

Thankfully, a colleague offered advice for teaching mathematics in this new digital environment. After his recommendation, I bought a tablet. While I awaited the tablet's arrival, I scoured through many high-quality instructional YouTube videos on mathematics. Their creativity and enthusiasm fueled my hope that the beauty and utility of mathematics could still be appreciated by students, only this time enhanced by technology. Of those YouTube channels, Eddie Woo's and Po-Shen Loh's stood out to me. The high standard of mathematics instruction set by these two renowned educators with their colorful and easy-to-look-at mathematical diagrams drove me to do the same. I could not wait to employ this technologically charged way of exchanging and presenting mathematical ideas with my students.

The use of the tablet for my teaching became essential. The GoodNotes application was also an indispensable tool that streamlined the organization of my notes for my numerous classes, allowed me to produce high-quality PDF files, and easily synched with Google Classroom. I remember experiencing the joy of writing and displaying mathematics on the tablet by sharing it through Zoom. Over the past year, many students expressed that they enjoyed learning an array of topics in a remote environment, from drawing geometric figures to calculating the volumes of solids of revolution.

In light of the litany of hurdles that I continue to face daily after teaching remotely for over a year, I can say, unequivocally, that teaching mathematics is alive and well. Technology became a lifeline for my teaching. It was possible for me to provide an educational experience despite the turmoil associated with surviving a pandemic. As we head back into the classrooms, I look forward to using technology not merely as a supplementary educational tool but rather as one that has become inextricably linked to displaying the beauty and awe-inspiring power of mathematics.

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NOTES FROM THE FIELD

Navigating the Pandemic through Interdisciplinary Collaborations

Estefania Hereira
Queens College,
Flushing International High School

To alleviate how disoriented students and teachers felt navigating the isolating online environment, we merged our Mathematics, Computer Science, Media Arts, History, and English courses into two: Humanities and STEAM. The new interdisciplinary curricula allowed new pathways for student choice and ownership over their learning.

I co-designed four projects with my student teacher and the Media Arts teacher. For one of the projects, students engaged in reading music and composing songs through a trigonometry lens. They sketched the sine curves of selected musical notes and worked in groups to understand how the functions' parameters relate to sound. Students who focused on music for their culminating project designed an instrument using simple art supplies, household objects, Scratch coding, and Makey Makeys (an electronic circuit board that connects the keys of a computer to conductive materials). They explored how songs and instruments fulfill specific human needs like building connections, strengthening memory, relieving stress and anxiety, making information accessible, and sharing truths. They found images to create metaphorical sketches and physical prototypes that would convey a sociopolitical issue. After several iterations, one student designed a 6-key piano prototype out of cardboard, printed dollar bills, copper tape, and cardstock to express economic inequities. He cut the dollar bills in proportion to the 2019 median weekly salaries of native or foreign-born Black, LatinX, and White populations in the United States.

In a written reflection, one student shared that "Doing projects in STEAM taught me skills such as critical thinking and connecting to the real world. It's not just about solving numbers; it's understanding the language of math in different forms." Another shared, "It works so good. You are taking the different subjects at the same time and crazily finding a way to combine them. The projects are not only used once and are just there to pass. They become drafts for bigger projects. They have a meaning–change your society, make it better, and find a solution to delete injustices we have been facing."

The devastating reality of COVID forced us to release all preconceived notions and depend on the people around us. My team and I created meaningful educational experiences in which our students had a say. My instruction was headed in an interdisciplinary direction for many years, but the pandemic pushed me to explore this approach. Some may perceive this year as an educational "failure," but it was more like what my student said about STEAM: "It's kinda hard at the beginning because it's pretty innovative. But we overcome it, and when all the pieces start to come together, it is very satisfying."

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NOTES FROM THE FIELD

Meeting Students Where They Are: A Schoolteacher's Brief Account of Teaching in the Pandemic

Brian Darrow, Jr.
Teachers College, Columbia University

On March 12, 2020, minutes before the end of the day, the principal made the announcement. The students cheered throughout the school as the news came through the intercom. That afternoon, we all left behind what would become known as "how school used to be."

For mathematics teachers, this meant redefining practice in a discipline where many pedagogical principles are built upon tangibles, such as pencil and paper, and in-person, in-the-moment discourse. Mathematics education, therefore, is particularly vulnerable to many of the challenges associated with teaching students at a distance. Although my colleagues and I continue to overcome these challenges to produce high-quality mathematical experiences for our students, we have consistently struggled to engage them. I have always been able to generate engagement through leveraging the genuine, positive relationships that I have with my students. This year, however, both developing these relationships and leveraging them was more difficult than ever.

I realized early in the year that school was the least of many of my students' worries. They were facing overwhelming, life-altering circumstances in their personal lives. Many teachers I know attempted to maintain the same standards of rigor and integrity as they have in previous years. I respected their approach to make the school experience as familiar as possible to students, and I observed many of them respond positively to this practice. However, based upon what I was seeing with my students, I knew I had to scale back many of the expectations and standards that I have required in the past. I had to focus instead on meeting each student where they were both academically and personally. More than ever before, I had to extend my efforts beyond the school day to negotiate work completion; make exceptions; create alternative assessments and assignments; and establish and maintain contact with students and their families. Although my hard work, flexibility, and generosity were often taken advantage of, my attempt to put compassion and empathy first was not made in vain. Students and their families have consistently reported to me that this approach enabled me to reach students who were in danger of personal crisis and nearing complete academic foreclosure. These conversations consistently reminded me that my efforts, and the efforts of my colleagues, were inextricably linked to my students' quality of life.

In reflecting on this year, I realized that there was no relative degree of success. Particular value judgments of teacher effectiveness and traditional metrics of student performance do not suffice. From my perspective, what matters is that the teachers invested more of themselves in their work this year than ever before. I am personally overwhelmed by how admirably and dutifully my colleagues and I responded to the challenges we faced this year. To me, our work this year serves as a symbol of our resilience and commitment to our calling.

ABOUT THE AUTHORS

Courtney K. Baker is an Assistant Professor of Mathematics Education Leadership at George Mason University. Her research interests include developing effective mathematics specialists and leaders that influence teaching and learning.

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Brian Darrow, Jr. is a high school mathematics teacher and a doctoral candidate in mathematics education at Teachers College, Columbia University. He recently served as Guest Editor for the *Journal of Mathematics Education at Teachers College* for the tenth and eleventh anniversary issues. Mr. Darrow is a career educator who has formerly held research and teaching positions at several academic institutions. His current research interests include combinatorial design theory, mathematical problem solving, and inhibitory control.

Justin Dimmel is an assistant professor of mathematics education and instructional technology at the University of Maine. Dimmel completed both an MS in mathematics (2013) and a Ph.D. in mathematics education (2015) at the University of Michigan. Prior to pursuing his graduate degrees, Dimmel worked for five years as a mathematics educator and school administrator at independent, adventure-based boarding schools in Massachusetts (The Shackleton School) and the Bahamas (The Island School), where he gained experience with place-based education. At the University of Maine, Dimmel leads the immersive mathematics in rendered environments (IMRE) laboratory. His recent work investigates student interactions with diagrams that are inscribed in immersive spaces.

Courtney Ferrell is a math teacher at Bronx Theatre High School in New York City, with Mathematics Education degrees from Marist College and Teachers College at Columbia and a School Building Leadership certificate from Hunter College. She is a Math for America Master Teacher Fellow. Her teaching interests include integrating kinesthetic, visual, design, theatrical, and real-world applications into the math classroom

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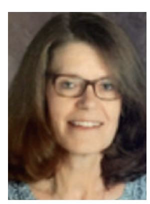

Terrie M. Galanti is an Assistant Professor of Secondary Mathematics and STEM Integration/Computational Thinking at the University of North Florida. Her research focuses on hyper-acceleration of Algebra I and opportunities for mathematical reasoning and sense-making with computational thinking.

Christopher Hanusa is a Professor of Mathematics at Queens College of the City University of New York. His mathematical research is in algebraic and enumerative combinatorics with over 25 peer-reviewed publications. He is also an internationally exhibited mathematical artist and an entrepreneur who started Hanusa Design, a 3D printed mathematical jewelry company.

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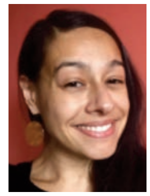
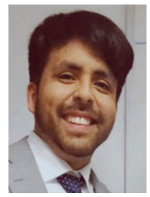
Peter Liljedahl is a Professor of Mathematics Education in the Faculty of Education. He is the former president of the International Group for the Psychology of Mathematics Education (PME), the current president of the Canadian Mathematics Education Study Group (CMESG), senior editor for the International Journal of Science and Mathematics Education (IJSME), on the editorial boards of ESM, JMTE, MERJ, MTL, CJSMTE, and a member of the NCTM Research Committee. Peter is a former high school mathematics teacher who has kept his research interest and activities close to the classroom. He consults regularly with teachers, schools, school districts, and ministries of education on issues of teaching and learning, problem solving, assessment, and numeracy.

Michelle Longhitano is a mathematics teacher and instructional coach at a diverse suburban high school in Westchester County, NY. There, she teaches a range of mathematics courses and runs professional development workshops that promote active and collaborative learning, with an emphasis on inclusive education and social-emotional learning. Michelle is a doctoral student in Mathematics Education at Teachers College, Columbia University. Currently, she is working on her dissertation research, where the focus is on using lesson study to develop an algebra lab model with high school teachers.



Nasriah Morrison is a doctoral student in Mathematics Education at Teachers College, Columbia University, as well as a Math for America Master Teacher Fellow. She has taught middle and high school mathematics in the New York City Department of Education for the past decade. Her research interests include the development of positive mathematics identities among historically underrepresented students in STEM.



Bryan Nevarez is both an adjunct lecturer at Queens College, City University of New York and a mathematics instructor at Think & Write, a private education center for grades K-12. He is a Queens College alum and earned his M.S. in Applied & Interdisciplinary Mathematics from the University of Michigan-Ann Arbor. He also serves on the Editorial Board for the AMC 8. His interests include the drafting of problems for mathematics competitions, the integration of technology in mathematics instruction and the impact of mathematical training upon student performance.

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Sian Zelbo teaches middle and high school mathematics at the Brearley School in New York City. She is also an adjunct faculty member at Stern College for Women, Yeshiva University, in Manhattan, where she teaches mathematics education. Dr. Zelbo received a J.D. from the University of Texas School of Law and an M.A. / Ph.D. in Mathematics Education from Teachers College, Columbia University. Dr. Zelbo's research interests include the history of American mathematics education and the history of recreational mathematics.

ACKNOWLEDGEMENT OF REVIEWERS

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Dyanne Baptiste
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Anne Uglum
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The Brearley School

JOURNAL OF MATHEMATICS EDUCATION AT TEACHERS COLLEGE

CALL FOR PAPERS

This call for papers is an invitation to mathematics education professionals, especially Teachers College students, alumni, and associates, to submit articles describing research, experiments, projects, innovations, or practices in mathematics education. The journal features full reports (approximately 3500 to 4500 words) and short reports (approximately 500 to 1500 words). Full reports describe findings from specific research, experiments, projects, innovations, or practices that contribute to advancing scholarly knowledge in mathematics education. Short reports (previously "Notes from the field") provide examples, commentary, and/or dialogue about practices out in the field of mathematics education or mathematics teacher education; examples from classroom experience are encouraged. Although past issues of *JMETC* focused around a theme, authors are encouraged to submit articles related to any current topic in mathematics education, from which pertinent themes for future issues may be developed. Articles must not have been submitted to or accepted for publication elsewhere. All manuscripts must include an abstract (approximately 150 words in length) and keywords. Manuscripts should be composed in Microsoft Word and follow APA format. Guest editors will send submitted articles to the review panel and facilitate the blind peer-review process. Articles for consideration should be submitted online at jmetc.columbia.edu, and are reviewed on a rolling basis; however, to be considered for the Fall issue, articles should be received by **September 1, 2021**.

CALL FOR REVIEWERS

This call for reviewers is an invitation to mathematics educators with experience in reading or writing professional papers to join the review panel for future issues of *JMETC.* Reviewers are expected to complete assigned reviews within three weeks of receipt of the manuscript in order to expedite the publication process. Reviewers are responsible for editorial suggestions, fact and citations review, and identification of similar works that may be helpful to contributors whose submissions appear appropriate for publication. Neither authors' nor reviewers' names and affiliations will be shared with one another; however, reviewers' comments may be sent to contributors of manuscripts to guide revision of manuscripts (without identifying the reviewer). If you wish to be considered for review assignments, please register and indicate your willingness to serve as a reviewer on the journal's website: jmetc.columbia.edu.

CALL FOR EDITOR NOMINATIONS

Do you know someone who would be a good candidate to serve as a guest editor of a future issue of *JMETC?* Students in the Program in Mathematics Education at Teachers College are invited to nominate (self-nominations accepted) current doctoral students for this position. Being asked to serve as a guest editor is a testament to the high quality and standards of the student's work and research. In particular, nominations for a guest editor should be a current doctoral student whose scholarship is of the highest quality, whose writing skills are appropriate for editorial oversight, and whose dedication and responsibility will ensure timely publication of the journal issues. All nominations should be submitted to Ms. Juliana Fullon at jmf2213@tc.columbia.edu .